Micro-Macro Duality
for Inductions/ Reductions

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Research Origin for Dressed Photon

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Abstract

Paradoxical appearance of negative metrics in the processes of emergences will be analyzed from the viewpoint of Morse theory, induced representations and of imprimitivity systems.

1 How to control inclusion relations

The essence of the present notes is to discuss the following issues which have been triggered by the requirement for a theoretical framework to treat Dressed Photons:

how to fill the gap between Macroscopic Phenomena & Microscopic Theory,

on the basis of Micro-Macro Duality in Quadrality Scheme,

comined with Saddle-Point Instability,

Through the following examples, Lorentz symmetry/ Regge structure/ Dressed photons/ Coulomb modes/ Tomita-Takesaki modular theory for statistical mechnaics, controlling mechanism will be explained on the basis of induced representations.

Existence of quantum modes with "**indefinite metric**" breaks the consistency of theory at Micro level, as is well known by the difficulties caused by **longitudinal photons**. Therefore, one always tries to avoid longitudinal photons in QED.

However, this is in contradiction to the existence of Coulomb modes in Macro world!!

To understand such contradictory situations, we need first re-examine the concept and phenomena of symmetry breaking.

2 Symmetry Breaking creates Symmetric Space

When symmetry of the system described by a group G is broken up to unbroken subgroup H, a homogeneous space G/H emerges in *sector classifying space*. In this situation, G/H is shown to be a **symmetric space** with many nice properties [Hel], according to the following criterion for *symmetry breaking*.

For this reason, **induced representation** Ind_H^G [Mackey] to describe the broken symmetry G on the sector classifying space G/H as a symmetric space has a strong connection with *automorphic forms* and *zeta functions* playing important roles in number theory.

The mutual relation between the quadrality scheme and the groups to describe symmetries can be depicted as follows:

	Spec= ***classifying space***	$=$	G/H	: Visible Macro
Emergence $\nearrow$	$\upharpoonleft\downharpoonright$		$\uparrow$	
(Family of) States	$\leftrightarrows$ ***Algebra***	$\curvearrowleft$	G	: Micro-Macro boundaries : t-**ch**
	$\upharpoonleft\downharpoonright$ $\nearrow$		$\uparrow$	
	Dynamics	$\curvearrowleft$	H	: Invisible Micro: s-**ch**

2.1 Symmetry Breaking

General definition of symmetry breaking [IO03]:

Definition (*Symmetry Breaking*): Let $\mathcal{X}$ be a C*-algebra describing quantum fields with an automorphic action $\tau{:}\mathcal{X} \underset{\tau}{\curvearrowleft} G$ of a Lie group G on $\mathcal{X}$ and $(\pi, \mathfrak{H})$ be a representation of $\mathcal{X}$. If the spectrum $Spec(\mathfrak{Z}_\pi(\mathcal{X}))$ of its center $\mathfrak{Z}_\pi(\mathcal{X}) = \mathfrak{Z}(\pi(\mathcal{X})'')$ is pointwise G-invariant (almost everywhere w.r.t. the central measure), the symmetry (G, τ) on $\mathcal{X}$ is said to be ***unbroken*** in $(\pi, \mathfrak{H})$ and ***broken*** otherwise.

The reason for complicated situations concerning symmetry breaking in QFT is due to such a contrast between quantum systems with *finite* vs. *infinite* degrees of freedom: while the use of a unitary representation U of G leads automatically to the ***unbroken*** symmetry (which is always the case for systems with *finite* degrees of freedom), the very non-existence of U realizable only in those with *infinite* degrees of freedom characterizes the ***broken*** symmetry. This is the reason why we need G-actions both in C*- and W*-versions in the above criterion for symmetry breaking.

2.2 Induced Representation from Unbroken to Broken

To streamline the discussion, we define "***augmented algebra***" [IO03] by a (C*-)crossed product $\mathcal{X} \rtimes \widehat{(H\backslash G)} =: \widehat{\mathcal{X}}$ of $\mathcal{X}$ with the dual $\widehat{(H\backslash G)}$ of (G/H), which allows unitary implementation of broken G at the expense of non-trivial center $\mathfrak{Z}_\pi(\widehat{\mathcal{X}})$ with

$Spec(\mathfrak{Z}_{\pi}(\widehat{\mathcal{X}})) = G/H$ in the representation of $\widehat{\mathcal{X}}$. Thus the corresponding von Neumann algebra $\pi(\mathcal{X})''$ can be taken as $(\pi \rtimes U_{\tau})(\widehat{\mathcal{X}})''$ in the above definition. The existence of a central spectrum as $Spec(\mathfrak{Z}_{\pi}(\widehat{\mathcal{X}})) = G/H$ suggests relevance of ***induced representations*** and ***imprimitivity*** [Mackey] involving the following ***exact sequences***:

$$\mathrm{Rep}(G/H) \hookrightarrow \mathrm{Rep}(G) \overset{Ind_H^G}{\leftrightarrows} \mathrm{Rep}(H),$$

$$H \underset{(Ind_H^G)^*}{\rightleftarrows} G \twoheadrightarrow G/H.$$

The bigger group G can be viewed as a principal H-bundle over base space $G/H = Spec(\mathfrak{Z}_{\pi}(\widehat{\mathcal{X}}))$ as sector classifying space, and dual map $(Ind_H^G)^*$ of Ind_H^G (sometimes called "Wigner rotation") plays the role of ***gauge connection***.

2.3 Physical Meaning of Central Spectrum

Note here that the starting point of our present discussion is just a C*-dynamical system $\mathcal{X} \underset{\tau}{\curvearrowleft} G$ given by C*-algebra $\mathcal{X}$ of quantum fields acted upon by a Lie group G of the symmetry of the system. In a sense, however, spacetime background.of the dynamical system $\mathcal{X} \underset{\tau}{\curvearrowleft} G$ without being mentioned at the beginning, has emerged automatically in the form of G/H as a result of the symmetry breaking from G to H. In this sense, the essence of symmetry breaking is crucial and universal for general understanding of the meaning of the above quadrality scheme for Micro-Macro duality.

For this purpose, we remark first such a crucial point that the center of represented algebra $\pi(\widehat{\mathcal{X}})''$ consisting of Macro variables of the system as low energy modes has such a spectrum as $Spec(\mathfrak{Z}_{\pi}(\widehat{\mathcal{X}})) = G/H$. Its non-trivial motion is driven by the action of G to exhibit the essence of symmetry breaking as the "infrared instability". Arbitrary representations of $\widehat{\mathcal{X}}$ are decomposed into the direct sum of G-*unbroken* factor representations and G-centrally ergodic non-factor representations (the latter ones corresponding to symmetry breaking). according to which a "phase diagram" can be drawn on the central spectrum.

2.4 Symmetry Breaking and Symmetric Spaces

Symmetry Breaking of Lie group G with Lie algebra $\mathfrak{g}$ creates an interesting Micro-Macro interface between Micro level invariant under unbroken Lie subgroup H with Lie algebra $\mathfrak{h}$ and visible Macro level of sector classifying space $M = G/H$.

M: formed in the emergence of condensed order parameters which parametrize the so-called "degenerate vacua" arising from symmetry breaking.

According to the criterion for symmetry breaking, $M = G/H$ becomes a ***symmetric space*** (É. Cartan) [Hel] whose Lie structure $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ is characterized locally by the relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ [RIMS2014].

Here commutator $[\mathfrak{m}, \mathfrak{m}]$ of tangent vectors in M describes **holonomy effect** of the curvature of M in loop motions on M. Since a trajectory forming a loop returns to its starting point on sector classifying space M, net effect of the loop reduces to such components of transformation group as fixing the sector unchanged, being contained in unbroken symmetry corresponding to $\mathfrak{h}$, which can be expressed as Macro loops $[\mathfrak{m}, \mathfrak{m}]$ penetrated by Micro arrows in $\mathfrak{h}$.

2.5 Examples of Symmetric Spaces: Chiral symmetry, Lorentz boosts & Second Law of Thermodynamics

1) Typical example of symmetry breaking yielding symmetric space structure can be found in chiral symmetry of current algebra:

$$[V, V] = V, [V, A] = A, [A, A] = V,$$
$$(V \in \mathfrak{h}\text{: vector currents, } A \in \mathfrak{m}\text{: axial currents}).$$

2) For Lorentz group $L_+^{\uparrow}$ as G with rotation group $SO(3)$ as unbroken H, we can find a symmetric space $M = G/H \cong \mathbb{R}^3$ given by the space of all Lorentz frames connected by Lorentz boosts. In fact, relations $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ with $\mathfrak{h} := \{M_{ij}; i, j = 1, 2, 3, i < j\}$, $\mathfrak{m} := \{M_{0i}; i = 1, 2, 3\}$ can be extracted from the Lorentz Lie algebra:

$$[iM_{\mu\nu}, iM_{\rho_\sigma}] = -(\eta_{\nu\rho} iM_{\mu\sigma} - \eta_{\nu\sigma} iM_{\mu\rho} - \eta_{\mu\rho} iM_{\nu\sigma} + \eta_{\mu\sigma} iM_{\nu\rho}).$$

3) The essence of characterization of symmetric spaces by "Macro loops $[\mathfrak{m}, \mathfrak{m}]$ penetrated by Micro arrows in $\mathfrak{h}$" can be exhibited directly in Macro world in the form of ***second law of thermodynamics***. Its mathematical essense can be seen in the following exact sequence[1]

$$\Delta' Q \overset{q}{\hookrightarrow} \Delta E = \Delta' Q + \Delta' W \overset{p}{\twoheadrightarrow} \Delta' W,$$

$$\text{i.e., } \operatorname{Im}(q) = \ker(p),$$

This is the same as the relation $\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ to characterize Lie structure of homogeneous space $M = G/H$.

The cyclic processes of a heat engine correspond to loops on the thermodynamic phase space M described by thermodynamic variables and holonomy $[\mathfrak{m}, \mathfrak{m}]$ associated with such cycles describes the incoming & outgoing heat between the heat engine & the external world in combination with the relations $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and $\Delta E = \Delta' Q + \Delta' W = 0$: $-\Delta' W = -[\mathfrak{m}, \mathfrak{m}] = \Delta' Q > 0$, in which the characterization of M as a symmetric space $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ corresponds to the ***second law of thermodynamics*** in ***Kelvin's version!***

[1] Equality $\operatorname{Im}(q) = \ker(p)$ menas that the vanishing energy balance ($\ker(p)$) taken as the visible work is equivalent to the input-output of the heat ($\operatorname{Im}(q)$).

3 Sector Bundle & Holonomy

In the case of symmetry breaking of G up to its unbroken compact subgroup H, the sector structure should be understood in two levels, one with the totality $\widehat{H}$ of irreducible rep.'s of unbroken subgroup H of G, and the other with G/H as the broken part of G. To unify these two levels, it is convenient to introduce the concept of a sector bundle:

$$\widehat{H} \hookrightarrow G \underset{H}{\times} \hat{H} \twoheadrightarrow G/H.$$

In this context we can see the physical origin of space-time concept in its ***physical emergence process*** [IO10].

For simplicity, we assume here that a group G of broken internal symmetry be extended by a group $\mathcal{R}$ of space-time symmetry (typically translations) into a larger group $\Gamma = \mathcal{R} \dot{\times} G$ defined by a semi-direct product of $\mathcal{R}$ & G with $\Gamma/G = \mathcal{R}$.

In this case, the sector bundles have a double fibration structure:

$$\begin{array}{ccccc} \widehat{H} & \hookrightarrow & G \underset{H}{\times} \hat{H} & \hookrightarrow & \Gamma \underset{G}{\times} (G \underset{H}{\times} \hat{H}) = \Gamma \underset{H}{\times} \hat{H} \\ & & \downarrow & & \downarrow \\ & & G/H & & \Gamma/G = \mathcal{R} \end{array} .$$

3.1 Holonomy along Goldstone condensates

Thus, we see that Spec= sector-classifying space has three different axes on different levels:

i) sectors $\widehat{H}$ of ***unbroken*** symmetry H,

ii) degenerate vacua $G/H = M$ due to ***broken internal*** symmetry [IO03, IO04],

iii) $\Gamma/G = \mathcal{R}$ as emergent ***space-time*** [IO10] in broken external symmetry.

These axes appear geometrically as a series of structure group contractions $H \leftarrow G \leftarrow \Gamma$ of principal bundles $P_H \hookrightarrow P_G \hookrightarrow P_\Gamma$ over $\mathcal{R}$, specified by ***solderings*** as bundle sections, $\mathcal{R} \overset{\rho}{\hookrightarrow} P_G/H = P_H \underset{H}{\times} (G/H)$, $\mathcal{R} \overset{\tau}{\hookrightarrow} P_\Gamma/G = P_G \underset{G}{\times} (\Gamma/G) = P_G \underset{G}{\times} \mathcal{R}$, which correspond physically to ***Goldstone modes***.

3.2 Helgason duality with Hecke algebra

We see the duality between ***Helgason duality*** [HRad] $K\backslash G \leftrightarrow G/H$ in
$$\begin{array}{ccccc} & \nearrow & K\backslash G/H & \nwarrow & \\ K\backslash G & & \leftrightarrow & & G/H \\ & \nwarrow & G & \nearrow & \end{array}$$
with Radon transform & ***Hecke algebra*** $K\backslash G/H$

and the algebraic structure of "***augmented algebras***" [IO03] for symmetry breaking as "stereo-graphic" extension of planar diagrams:

$$
\begin{array}{ccccc}
{}_{G/H}\swarrow & \mathcal{X}^H = \widetilde{\mathcal{X}}^G & \searrow_H \\
\swarrow & \Downarrow & \searrow \\
\widetilde{\mathcal{X}}^H & \Downarrow & \mathcal{X} \\
\downarrow_H & \searrow\!\searrow \quad \swarrow_{G/H} & \downarrow \\
\downarrow & \widetilde{\mathcal{X}} & \downarrow \\
\downarrow \; \swarrow & \Downarrow & \searrow\!\searrow \; \downarrow \\
\widehat{H\backslash G} & \hookrightarrow \; \widehat{G} \; \twoheadrightarrow & \widehat{H}
\end{array}
\quad \rightleftarrows \quad
\begin{array}{ccc}
{}_{\mathcal{R}}\swarrow & \mathcal{O}_\rho = \mathcal{O}_d^H & \searrow_H \\
\swarrow & \Downarrow & \searrow \\
\mathcal{A}(\mathcal{R}) & \Downarrow & \mathcal{O}_d \\
\downarrow_H & \searrow\!\searrow \quad \swarrow_{\mathcal{R}} & \downarrow \\
\downarrow & \mathcal{X}(\mathcal{R}) & \downarrow \\
\downarrow \; \swarrow & \Downarrow & \searrow\!\searrow \; \downarrow \\
\widehat{\mathcal{R}} & \hookrightarrow \; \widehat{\Gamma} \; \twoheadrightarrow & \widehat{H}
\end{array}
\quad . \quad
\begin{array}{c}
\text{[same kinds of} \\
\text{lines constitute} \\
\text{exact sequences]}
\end{array}
$$

Similar push-out diagram appears also in Doplicher-Roberts reconstruction [DR90] for field algebra $\mathcal{X}(\mathcal{R})$ with unbroken symmetry

3.3 Symmetric space structure = Maxwell-type equation due to symmetry breaking

Symmetric space structures of $G/H = M$ & $\Gamma/G = \mathcal{R}$ arising from symmetry breaking are characterized by the equation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ to ***connect holonomy*** $[\mathfrak{m}, \mathfrak{m}]$ (in terms of curvature) ***with unbroken generators*** in $\mathfrak{h}$.

It is really interesting to note that ***this feature is shared in common by Maxwell & Einstein equations*** of electromagnetism and of gravity, respectively:

$$\text{LHS: (curvature } F_{\mu\nu} \text{ or } R_{\mu\nu}) = (\text{source current } J_\mu \text{ or } T_{\mu\nu}) : \text{RHS},$$

which can be seen by noting that all the quantities $[\mathfrak{m}, \mathfrak{m}]$, $F_{\mu\nu}$ and $R_{\mu\nu}$ on LHS represent holonomy terms and that those on RHS are associated with generators $\mathfrak{h}$ of unbroken subgroups.

In the usual context (related to the 2nd Noether thm), Maxwell equation is understood as an identity following from the gauge invariance of "action integral" under local gauge transformations. In contrast we have *no such classical quantities as action integrals nor Lagrangian densities* defined in our algebraic & categorical formulation of quantum fields.

3.4 Possibility for Dressed Photon equations?

Without such quantities as "action integrals", symmetry breaking criterion with $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ tells us that Maxwell-type equation with curvature term $[\mathfrak{m}, \mathfrak{m}]$ on the left-hand side and the internal symmetry term $\mathfrak{h}$ on the right-hand side is just a consequence of symmetry breaking of local gauge invariance into spacetime and internal symmetries. Putting the Clebsch-dual electromagnetic field $S_{\mu\nu}$ due to Sakuma [SOO] in the place of $[\mathfrak{m}, \mathfrak{m}]$, therefore we can learn that $S_{\mu\nu}$ represents the condensation effect of dressed photons.

3.5 Galois Functor in Doplicher-Roberts reconstruction of symmetry

We recall here how Doplicher & Roberts (DR) [DR90] recovers internal symmetry group from *DR category* $\mathcal{T}$ of local excitations as *group-invariant* data.

Objects of $\mathcal{T}$: local endomorphisms $\rho \in End(\mathcal{A})$ of observable algebra $\mathcal{A}$, selected by DHR localization criterion [DHR] $\pi_0 \circ \rho \restriction_{\mathcal{A}(\mathcal{O}')} \cong \pi_0 \restriction_{\mathcal{A}(\mathcal{O}')}$, and
Morphisms of $\mathcal{T}$: $T \in \mathcal{T}(\rho \to \sigma) \subset \mathcal{A}$ intertwining $\rho, \sigma \in \mathcal{T}$: $T\rho(A) = \sigma(A)T$.

The group H of unbroken internal symmetry arises as the group $H = End_{\otimes}(V)$ of unitary tensorial (=monoidal) natural transformations $u : V \to V$ with the representation functor $V : \mathcal{T} \hookrightarrow Hilb$ to embed $\mathcal{T}$ into the Hilbert-space category $Hilb$ with morphisms as bounded linear maps.

3.6 Galois Functor in Category & local gauge invariance

Recall that a natural transformation $u : V \to V$ is characterized by the commutativity diagrams:
$$\begin{array}{ccc} V(\rho) & \overset{u_\rho}{\to} & V(\rho) \\ V(T)\downarrow & \circlearrowleft & \downarrow V(T) \\ V(\sigma) & \underset{u_\sigma}{\to} & V(\sigma) \end{array}$$
, namely, $V(T)u_\rho = u_\sigma V(T)$ for $T \in \mathcal{T}(\rho \to \sigma)$.

Our simple proposal here is to define a local gauge transformation $\tau_u(V)$ of functor V by $\tau_u(V)(T) := u_\sigma V(T) u_\rho^{-1}$ corresponding to a natural transformation $u \in H = End_{\otimes}(V)$ [RIMS2013, RIMS2014].

Then, the above equality, $V(T)u_\rho = u_\sigma V(T)$, can be reinterpreted as *local gauge invariance* $\tau_u(V) = V$ of functor V under *local gauge transformation* $V \to \tau_u(V)$ induced by a natural transformation $u \in H = End_{\otimes}(V)$, as has been visualized in the context of lattice gauge theory.

4 *Trinity relation of Saddle point, Indefinte metric & Non-compact group*

For the purpose of theorteical description of dressed photons, crucial step will be to recognize proper dynamic functions in close relation with "tapering" cone structure formed by condensed dressed photons. To implement ideas in this direction, it is important to install the *Clebsch-dual variables* due to Sakuma [SOO] which carry *spacelike momenta* and constitute the characteristic *off-shell structure* of electromagnetic field.

To see the general meaning of off-shell structures, a ***trinity connection*** is to be focused, among ***saddle-point instability***, presence of ***indefinite metric*** (in some Hessians of *Morse functions*) and the action of a ***non-compact group*** on the saddle point.

In wider contexts including thermodynamics, statistical mechanics, gauge theories and induced representations of groups, most important common aspects are the trinity connection between ***saddle points*** & ***indefinite metric***, due to the co-existence of ***stable & unstable*** directions corresponding to compact subgroup H and to ***non-compact*** G/H part of the bigger group G, respectively.

4.1 Saddle points and Morse theory

When this mechanism for determining geometric invariants is applied to sector classifying space, non-trivial relations between quantum Micro dynamics & geometric Macro structure of classifying space can be envisaged and described in terms of ***unstable modes*** and indefinite metric corresponding to saddle point structures. In *Morse theory* contexts [Morse] of deriving homologies and/or cohomologies as geometric invariants, they are determined by negative-metric components of ***Hessians*** defined as the second derivatives of ***Morse functions*** whose dimensionality is called "***Morse index***".

In concrete systematic descriptions of dynamical processes from this viewpoint, the actual meaning of treating "stability" aspects would be restricted to examining which "branches" would satisfy the (conditional) stability and which conditions can support the classifying space *Spec* describing the multi-sector structure serves as the setting up for such discussions.

4.2 Stability vs. instability

Thus it becomes possible for us to envisage the problems of whether stable or unstable naturally in wider perspectives. Moreover, this kind of contexts would require us to pursue such processes as the formation of classifying spaces *Spec* through emergences triggered by the instability at saddle points as the bifurcation points between stability & instability.

Through this kind of changes, big transitions would perhaps be implemented to enable us to be faithful to such natural recognition that ***dynamical motions are absolute and fundamental*** and ***stable states are conditional***.

Are the basic points for this direction hidden in "indefinite metric" which has been disliked so far?: answer to this question is really affirmative when we combine the following points, i) ***indefinite metric at the saddle point***, ii) ***symmetry breaking aspects inherent in Maxwell equation***, and iii) ***spacelike supports of dressed photon momenta described by Clebsch-dual field***.

4.3 Roles Separated into Micro vs. Macro with geometric invariants

Now we consider the problems along the above line.

For this purpose, we consider first 1) induced representation of groups, and 2) guage theories.

1) As is well known,

Lie group G: compact $\Longleftrightarrow$ Killing form θ of its Lie algebra $\mathfrak{g}$ is negative definite,

G: non-compact $\Longleftrightarrow$ Killing form θ of $\mathfrak{g}$ is indefinite

While irreducible representation (σ, W) of maximally compact subgroup H is realized in a (finite-dimensional) positive definite Hilbert space W, the irreducible finite-dimensional representation of ***non-compact semisimple*** G is possible only in a vector space ***with indefinite metric***.

5 Induced representation Ind_H^G

In this situation, the induced representation $Ind_H^G(\sigma)$ [Mackey] of G induced from a representation (σ, W) of H can be realized in an *infinite-dimensional* positive definite Hilbert space $L^2_\sigma(G \to W) = L^2(G) \underset{H}{\otimes} W$ which is defined as the subspace of W-valued functions $\xi : G \longrightarrow W$ on G satisfying the condition of H-equivariance:

$$\xi(gh) = \sigma(h)\xi(g) \quad \text{for } g \in G \ \text{ and } h \in H.$$

According to the equivariance condition, the representation (σ, W) of H is recovered (by the left translation) at the origin $e \in G$:

$$[l_{h^{-1}}\xi](e) = \xi(he) = \xi(eh) = \sigma(h)\xi(e).$$

In this way the appearance of ***indefinite metric*** in the representation space due to non-compactness of G is ***absorbed into*** the ***infinite dimensionality*** of the representation space.

5.1 Micro-unphysical can become Macro-physical

2) In the case of (abelian) gauge theory with a gauge potential A_μ, its Lorentz covariant formulation is possible only in a state vector space with an indefinite metric. In the total space with indefinite metric, we can introduce the concept of a physical subspace $\mathcal{V}_{phys}$ consisting of gauge-invariant physical modes, by imposing such a "subsidiary condition" [KO] as $\Phi \in \mathcal{V}_{phys} \Longleftrightarrow (\partial_\mu A^\mu)^{(+)}\Phi = 0$. In this physical subspace $\mathcal{V}_{phys}$ longitudinal modes causing the difficulties of indefinite metric are shown to be absent, according to which consistency of the probabilistic interpretation is guaranteed within $\mathcal{V}_{phys}$ at the Micro level.

Existence of quantum modes with indefinite metric spoils the consistency of the theory at Micro levels, as is seen in the difficulties caused by longitudinal photons in probabilistic interpretation. For this reason, one tries to exclude longitudinal photons from QED and it is common wisdom that such unphysical modes can be systematically expelled from physical subspace of physical modes selected by imposing a suitable "subsidiary condition".

5.2 Coulomb mode as Micro-unphysical & Macro-physical

As a plain fact in real Macro world, Coulomb modes exist and mediate interactions between electric charges. According to the *standard* "quantum-classical correspondence", mutual relations between Micro & Macro, between quantum & classical, can be understood in such a way that quantum observables non-commutative in Micro scales become mutually commutative classical observables in the "classical limit" with $\hbar \to 0$ and that classical observables can be "quantized" through imposing the canonical commutation relations as a result of which quantum theory equipped with non-commutative quantum observables can be realized.

In non-trivial emergence processes to Macro, however, this simple-minded picture between quantum & classical observables fails to hold by such **paradoxical situations** that some physical variables invisible (or driven away as unphysical modes) at Micro level may become **visible in Macro world**, as is exemplified by longitudinal Coulomb modes. In such cases, how is the fate of risky "indefinite metric"??

5.3 How Induced Representations avoid Indefinite Metric?

In emergence to Macro, indefinite metric in Micro disappears to be substituted by geometric non-triviality. This phenomenon takes place also in the construction of representations of non-compact groups induced from its compact subgroup.

Typical example found in ∞-dimensional unitary rep. of (inhomogeneous) Lorentz group $(\mathbb{R}^4 \rtimes)SL(2,\mathbb{C})$, first established by a physicist E. Wigner in 1939 [Wig39] in use of the method of induced representations. In spite of non-compactness of $SL(2,\mathbb{C})$, we do not encounter infefinite metric in this situation.

Mechanism of induced representations to suppress infefinite metric can be seen in such a form that non-compact group $SL(2,\mathbb{C})$ possibly inducing infefinite metric is treated here as base space $M := G/H = SL(2,\mathbb{C})/SU(2)$ of $SU(2)$-bundle:

$$H := SU(2) \hookrightarrow G = SL(2,\mathbb{C}) \twoheadrightarrow M = SL(2,\mathbb{C})/SU(2).$$

5.4 Alternation between indefinite metric in Micro & geometric non-triviality in Macro

At each point of base space $M = SL(2,\mathbb{C})/SU(2)$ (as a part of sector classifying space), we have a fixed Lorentz frame acted upon by rotation group $SU(2)$ as the structure group of each Lorentz frame and the actions of Lorentz boosts $SL(2,\mathbb{C})$ are just to move from one Lorentz frame to another, which do not exhibit infefinite metric related with $SL(2,\mathbb{C})$ like the case of its matrix representation.

On this geometric setting up, the representation $Ind_{SU(2)}^{SL(2,\mathbb{C})}(\sigma) \in \mathrm{Rep}(SL(2,\mathbb{C}))$ induced from a representation $\sigma \in \mathrm{Rep}(SU(2))$ is defined on the Hilbert space $L^2_\sigma(SL(2,\mathbb{C}) \to W)$ as given above, which is isomorphic to $L^2(M) \otimes W$ in the present situation where the base space $M = SL(2,\mathbb{C})/SU(2)$ is a symmetric space.

5.5 "Wigner rotation" as Dual of Ind_H^G

Owing to the duality,

$$[Ind_H^G(\sigma)](g) = \langle g | Ind_H^G(\sigma)\rangle = \langle (Ind_H^G)^*(g)|\sigma\rangle = \sigma((Ind_H^G)^*(g)),$$

each group element $g \in G$ belonging to non-compact $G = SL(2, \mathbb{C})$ is transferred to $(Ind_H^G)^*(g)$ belonging to compact subgroup $H := SU(2)$:

$$\mathrm{Rep}(SU(2)) \ni \sigma \longmapsto Ind_H^G(\sigma) \in \mathrm{Rep}(SL(2, \mathbb{C})),$$

$$SU(2) \ni (Ind_H^G)^*(g) \leftarrow g \in SL(2, \mathbb{C}).$$

This mapping $(Ind_H^G)^*$ is called (in physics) "Wigner rotation", since each of its image $(Ind_H^G)^*(g) \in SU(2)$ is a rotation.

5.6 "Wigner rotation" as Gauge Connection

According to exact sequence $H \hookrightarrow G \twoheadrightarrow M = G/H$, group G can be interpreted as an H-principal bundle with structure group H over base space $M = G/H$. In this context, the sequences $\mathrm{Rep}(G/H) \hookrightarrow \mathrm{Rep}(G) \twoheadrightarrow \mathrm{Rep}(H)$ and $H \hookrightarrow G \twoheadrightarrow G/H$ are *split* exact sequences, owing to the induced representation $Ind_H^G : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G)$ and to the "Wigner rotation" as its dual $(Ind_H^G)^* : G \ni g \longmapsto (Ind_H^G)^*(g) \in H$, resepectively:

$$\mathrm{Rep}(G/H) \hookrightarrow \mathrm{Rep}(G) \overset{Ind_H^G}{\leftrightarrows} \mathrm{Rep}(H),$$

$$H \underset{(Ind_H^G)^*}{\rightleftarrows} G \twoheadrightarrow G/H.$$

I.e. vector bundle $\mathrm{Rep}(G)$ on base space $\mathrm{Rep}(H)$ with standard fiber $\mathrm{Rep}(G/H)$ has Ind_H^G as a horizontal lift.

Principal H-bundle G over G/H has a H-valued connection given by $(Ind_H^G)^*$.

$\Longrightarrow$ Induced representation gives a basis for structural analogy with gauge theory, in terms of gauge connection $(Ind_H^G)^*$ as a splitting of exact sequence.

5.7 No Problem for Macro Coulomb Mode

In the case of 2) with the Coulomb mode, we need not worry about the appearance of indefinite metric because the longitudinal Coulomb mode of classical gauge fields is already described in terms of the commutative variables. Instead, what can be non-trivial now is the possibility for condensed modes of particles due to Coulomb attractive force, according to which such non-trivial effects as superconductivity phenomena can be realized.

6 Spacelike momenta shared by statistical mechanics, Regge poles, dressed photons & Coulomb force

After the case studies of 1) induced representations and 2) gauge theories with Coulomb mode, what to be analyzed for the purpose of understanding common features among various composite systems with inclusion relations can be found as follows:

3) statistical mechanics and thermodynamics
4) Regge trajectories appearing in hadron scattering processes,
5) mechanism of dressed photons.

Because of the big difference in the appearance among these five cases, however, it may be unclear where we can find any coherent common features. Just skipping the detailed account along individual specific features, the common essence shared by all these cases can be found in the existence of the following three levels as well as their mutual relationship:

6.1 Exact Sequence consisting of Broken/ Unbroken Symmetry groups

a) a compact Lie group H to describe inivisible Micro dynamics associated with some flows,

b) the level of "*horizontal duality*" formed by the algebra $\mathcal{X}$ of observables to visualize H and the state space $E_{\mathcal{X}}(\subset \mathcal{X}^*)$ of $\mathcal{X}$ which is controlled by a Lie group G containing H as a subgroup, and,

c) the sector classifying space $Spec(\supset G/H)$ emerging from the states $E_{\mathcal{X}}$ of $\mathcal{X}$,

What is most important is such a situation that the group $G(\supset H)$ controlling the level b) of "*horizontal duality*" is a non-compact Lie group with a Killing form with indefinite signature, arising from the extention of the group H of Micro dynamics, characterized by the exact sequences:

$$H \hookrightarrow G \twoheadrightarrow G/H,$$

$$\mathrm{Rep}(G/H) \hookrightarrow \mathrm{Rep}(G) \twoheadrightarrow \mathrm{Rep}(H).$$

6.2 Examples of Broken/ Unbroken Sequences

For instance, in the case of dressed photons, the region with ***spacelike momenta*** is created by introducing the ***Clebsch-dual variables*** and in the case of Regge trajectory in hadron physics, the t and u***-channels*** formed via the duality transformations $s \rightleftarrows t$ & $s \rightleftarrows u$ interchanging s, t & u***-channels*** provide the stages of ***Regge trajectories*** consisting of the series of ***Regge poles with complex angular momenta.*** While well-known Gibbs formula $\langle A \rangle = Tr(Ae^{-\beta H})/Tr(e^{-\beta H})$ in statistical mechanics shows no remarkable structural features, it can be applied only to small finite systems with discrete energy spectrum, In contrast, ***Tomita-Takesaki modular theory*** required for the treatment of general systems with infinite degrees of freedom is equipped with such a double structure as consisting of the von Neumann algebra $\mathcal{M}$ of physical variables in the system and its modular dual $\mathcal{M}' = J\mathcal{M}J$ whose composite system $\mathcal{M} \vee J\mathcal{M}J$ is controlled by the Hamiltonian $H_\beta = -JH_\beta J$ with "indefinite metric", whose physical interpretation can be reduced to the concept of heat bath.

6.3 Induced Representations & Automorphic Forms

The induced representation $Ind_H^G(\sigma)$ of the Lorentz group $G = SL(2,\mathbb{C})$ determined by a unitary representation σ of the rotation group $H = SU(2)$ in a finite-dimensional vector space W is given in an infinite-dimensional Hilbert space V defined by

$$V := \{\varphi : G \longrightarrow W; \varphi(gh) = \sigma(h^{-1})\varphi(g) \text{ for } g \in G, h \in H\}$$

according to the defining equation $[Ind_H^G(\sigma)(g)\varphi](g_1) := \varphi(g^{-1}g_1)$, which reproduces $\sigma(h)$ for $h \in H$ at $g = e \in G$:

$$[Ind_H^G(\sigma)(h)\varphi](e) = \sigma(h)[\varphi(e)].$$

6.4 Automorphic Forms arising from Induced Representation

By means of the horizontal lift $G/H \longrightarrow G$ of $G/H = SL(2,\mathbb{C})/SU(2)$ associated with the "Wigner rotation" $(Ind_H^G)^*$, the domain of $Ind_H^G(\sigma)$ can be shifted from G to G/H. Therefore, if we express the elements $g \in G$ in the form of fractional linear transformation, the above definition of V can be rewritten with as

$$V = \{\varphi : G/H \to W; \varphi(gz) = \sigma(cz+d)^{-1}\varphi(\frac{az+b}{cz+d}),$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in G/H\},$$

which shows that the module V consists of automorphic forms φ. Since automorphic forms are transformed into ζ functions by Mellin transform, the pair (G,H) with G/H a symmetric space is related to the number-theoretical contexts.

6.5 Fractional Linear Transformaions

While the use of *fractional linear transformation:* $gz := \frac{az+b}{cz+d}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ may look accidental owing to the (2×2)-matricial form of $SL(2,\mathbb{C})$, this is not the case because this speciality can be easily lost by such identification of the Lorentz group as $G \simeq SO(1,3) \hookrightarrow M(4,\mathbb{R})$. Actually, what is essential is not such a special form of matrices but the decomposition of representation vector space $\mathfrak{V}$ of G into unbroken $\mathfrak{V}_1$

and broken subspaces $\mathfrak{V}_2$, $\mathfrak{V} = \mathfrak{V}_1 \oplus \mathfrak{V}_2$, according to which G has such a decomposition as $\begin{array}{c|cc} & \mathfrak{V}_1 & \mathfrak{V}_2 \\ \hline \mathfrak{V}_1 & A & B \\ \mathfrak{V}_2 & C & D \end{array}$ in a certain neighbourhood of the identity element of G.

6.6 Flag manifold as Generalization of Fractional Linearity

Moreover, if we want to extend the above bipolar contrast between unbroken vs broken into some scale-dependent multi-polar gradations of symmetry breakings along many steps, we can consider such a flag manifold structure as related with a multi-component decomposition $\mathfrak{V} = \mathfrak{V}_1 \oplus \mathfrak{V}_2 \oplus \cdots \oplus \mathfrak{V}_r$ of the representation space $\mathfrak{V}$:

$$\begin{aligned} &G = U(p_1 + p_2 + \cdots \ + p_r) \\ &\quad \curvearrowright G/H = U(p_1 + p_2 + \cdots \ + p_r)/[U(p_1) \times U(p_2) \times \cdots \ \times U(p_r)], \end{aligned}$$

which may be related with the continued fractions. In this context, we can see the intrinsic relation between fractional linearity and Grassmann manifold in the case of $r = 2$.

7 "Indefinite Metric" inherent in Modular Structure of Thermal Equilibrium

Here we want to touch on a blind spot in the "common sense" in physics which can interpret the "stability" of a state only in such a restricted form as the poisitivity of the energy in the form of spectral condition.

While, in inifinite system with the operator $e^{-\beta H}$ out of trace class, it is impossible to separate sharply the physical system and its heat bath, the mutual relation between them can be mathematically understood [HHW, BR] by the relation:

$$H_\beta = -JH_\beta J. \tag{1}$$

If the component H of H_β acting on the system $\mathcal{X}_\omega$ can safely be extracted and be separated from that on the commutant $\mathcal{X}'_\omega$, then the essential contents of this equation could be seen in such a form as

$$H_\beta = H - JHJ,$$

7.1 Negative Metric in Modular Theory and Heat Bath

In infinite systems, however, meaning of the above H is only formal. Apart from this subtlety, the above formal equation explains that anti-unitary operator J interchanges the system & its heat bath. Since total system consisting of the system & heat bath has Hamiltonian H_β whose spectrum is positive/ negative symmetric as in (1), negative energy component may be interpreted as energy going from the system to the heat bath. Interestingly enough, concept of "heat bath" which is mysterious but important in thermodynamics has once been expelled by Gibbs formula $\langle A\rangle = Tr(Ae^{-\beta H})/Tr(e^{-\beta H})$ (applicable only for the system with *discrete spectrum*), but, has survived.in the abstract form in algebraic general formulation of statistical mechanics based upon the Kubo-Martin-Schwinger condition [KMS, HHW, BR]:

$$\omega_\beta(AB(t)) = \omega_\beta(B(t-i\beta)A),$$

which is free from such a restriction of discrete energy spectrum.

Similarly to longitudinal photons with "***negagive metric***" Hamiltonian H_β of the total system contains negative component (formally $-JHJ$), which means the **existence of a saddle point instability** associated with thermal equilibrium states. Without unstable modes and their condensations, existence of Macro heat bath may have been impossible.

8 Frobenius Reciprocity

Two opposite directions are involved in induced representations, to expand $\sigma \in Rep_H$ of smaller H into that $Ind_H^G(\sigma) \in Rep_G$ of bigger G, and to identify a given $\gamma \in Rep_G$ of G as $\gamma = Ind_H^G(\sigma)$ induced from $\sigma \in Rep_H$ of H. This latter process is controlled by the **imprimitivity**. Mutual relation between two processes is controlled by **Frobenius reciprocity**:

$$Rep_H(\gamma \upharpoonright_H \longrightarrow \sigma) \leftrightarrows Rep_G(\gamma \longrightarrow Ind_H^G(\sigma))$$

or

$$Rep_G(Ind_H^G(\sigma) \longrightarrow \gamma) \leftrightarrows Rep_H(\sigma \longrightarrow \gamma \upharpoonright_H),$$

where $Rep_G(\gamma_1 \longrightarrow \gamma_2)$ means the set of intertwiners $T : \gamma_1 \longrightarrow \gamma_2$ from γ_1 to γ_2 satisfying the intertwining relation $\forall g \in G\ T\gamma_1(g) = \gamma_2(g)T$, namely,

$$T \in Rep_G(\gamma_1 \longrightarrow \gamma_2) \Longleftrightarrow \forall g \in G : T\gamma_1(g) = \gamma_2(g)T$$

$$\begin{array}{ccc} V_{\gamma_1} & \xrightarrow{T} & V_{\gamma_2} \\ \gamma_1(g)\downarrow & \circlearrowleft & \downarrow \gamma_2(g) \\ V_{\gamma_1} & \xrightarrow[T]{} & V_{\gamma_2} \end{array}$$

9 Towards Theory of Dressed Photons

In order to construct a consistent theory for describing dressed photons, it will become a crucial breakthrough to reproduce faithfully its proper dynamic functions by grasping properly the "tapering" cone structure formed by the condensed dressed photons. To implement the ideas in this direction, it is important to install the ***Clebsch-dual electromagnetic field*** [SOO] discovered by Sakuma carrying spacelike momenta which constitute the characteristic off-shell structure of electromagnetic field. which forms the Micro-Macro boundary level described by a symmetric space $G/H = Spec$ arising from a broken symmetry by visualizing the s-channel strictire at the invisible Micro level into spacelike t-channel.

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A Brief Summary of Micro-Macro Duality in Quadrality Scheme

Integrating [dynamical aspects of the system in question] with [geometric description of the relevant structure in terms of invariants generated by dynamical processes which implement classification of the processes and structures]

$\Longrightarrow$ category-theoretical framework of "Micro-Macro duality+quadrality scheme" ([IO03]; I.O., "Quantum Fields and Micro-Macro Duality" [IO13] [2013, in Japanese] and also see [IOOk13]) by incorporating categorically natural *duality between dynamical processes & classifying spaces.*

By analyzing closely in this framework dynamical processes and classifying scheme based on geometric invariants generated by the former processes, we can understand that both of invisible Micro domain corresponding to dynamical processes and of visible Macro structure to the classifying structure in terms of geometric invariants constitute duality structure, to be called "***Micro-Macro duality***" [IO06].

A.1 Quadrality Scheme

Duality between on-shell $\leftrightarrows$ off-shell means that on-shell corresponds to the particle-like Macro and the off-shell to the existence of quantum fields in virtual invisible modes.

Micro processes of motions can be described by a group(oid) structure acting on the algebras of physical quantities, Macro classifying structure emerging from dynamical processes can be extracted from the structure of state space as the dual of algebra of physical quantities and a geometric space emerges consisting of classifying indices extracted from states which functions as the dual of the Micro dynamical system. Putting altogether these four ingredients of dynamics, algebras, states and classifying space, they constitute a "***quadrality scheme***" describing "***Micro-Macro duality***" [IO06]:

$\nearrow$	***Classifying Space*** $=$ ***Spec***	
(*Family of*) ***States***	$\leftrightarrows$ $\upharpoonleft$ (***Representations***) $\downharpoonright$ $\leftrightarrows$	***Algebra*** .
	Dynamics	$\nearrow$

A.2 Emergence of sector classifying space

In this mathematical framework for describing emergence process, crucial roles are played by the concept of a "sector".

What is a ***sector***: for the mathematical description of a quantum system, we need a ***non-commutative*** (***C****-)***algebra*** $\mathcal{X}$ (: ***Algebra***) of physical variables to characterize the system and a certain family of ***states*** $\omega \in E_{\mathcal{X}}$ to quantify measured values $\omega(A)$ of physical variables $A \in \mathcal{X}$. According to GNS theorem [BR], a representation $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ (called GNS representation) of $\mathcal{X}$ is so constructed from ω that physical variables $A \in \mathcal{X}$ are represented as linear operators $\pi_\omega(A)$ acting on a Hilbert space $\mathfrak{H}_\omega$, the totality of which determines a very important concept of representation von Neumann algebra $\pi_\omega(\mathcal{X})'' =: \mathcal{X}_\omega$. Elements $C \in \mathfrak{Z}_\omega(\mathcal{X})$ of the center $\mathfrak{Z}_\omega(\mathcal{X})$ of $\mathcal{X}_\omega$ defined by

$$\mathfrak{Z}_\omega(\mathcal{X}) := \pi_\omega(\mathcal{X})'' \cap \pi_\omega(\mathcal{X})' = \mathcal{X}_\omega \cap \mathcal{X}_\omega',$$

are commuting with all elements X in $\mathcal{X}_\omega$: $[C, X] = 0$ for $\forall X \in \mathcal{X}_\omega$
and play the role of "order parameters" as commutative Macro observables.

A.3 Sectors = Factor States

Commutativity of center allows simultaneous diagonalization of $\mathfrak{Z}_\omega(\mathcal{X})$ yields spectral decomposition of a commutative algebra $\mathfrak{Z}_\omega(\mathcal{X}) = L^\infty(Spec)$ with spectrum of $\mathfrak{Z}_\omega(\mathcal{X})$ denoted by $Spec := Sp(\mathfrak{Z}_\omega(\mathcal{X}))$. The diagonalized situation with all the order parameters specified corresponds physically to a ***pure phase***, or mathematically corresponding to a quasi-equvalence class of a ***factor state*** γ with a trivial cener: $\mathfrak{Z}_\gamma(\mathcal{X}) = \mathcal{X}_\gamma'' \cap \mathcal{X}_\gamma' = \mathbb{C}1$ which is called a ***sector***. Here ***quasi-equvalence*** [Dix] means ***unitary equivalence up to multiplicity*** and a factor state corresponds to a minimal unit of states or representations in the sense that its center cannot be decomposed any more.

A.4 Sectors and Disjointness

To understand properly the concept of sectors, it is crucial to note the following points about the mutual relations between different sectors. Namely, the relation betwen two ***different sectors*** π_1, π_2 is expressed by the concept of disjointness as follows:

$$T\pi_1(A) = \pi_2(A)T \quad (\forall A \in \mathcal{X}) \implies T = 0,$$

which is stronger than unitary inequivalence and has deep implications as seen later. Macro quantities characterized by their commutativity appear as the center $\mathfrak{Z}_{\omega}(\mathcal{X})$ of a mixed phase algebra $\pi_{\omega}(\mathcal{X})'' = \mathcal{X}_{\omega}$ containing many different sectors as pure phases, and its spectrum $Spec = Sp(\mathfrak{Z}_{\omega}(\mathcal{X}))$ as realized values $\chi \in Spec$ of order parameters $C \in \mathfrak{Z}_{\omega}(\mathcal{X})$ discriminates the pure phases contained in the mixed phase state ω, The sectors as pure phases play the roles as the Mico-Macro boundary between quantum Micro system & classical Macro system as the environment, and they unify, at the same time, both these into a Micro-Macro composite system as a mixed phase.

A.5 Relations among Sectors

According to this story, the duality between intra-sectorial domains vs. inter-sectorial relations holds as follows:

$\longleftarrow$ ***Visible***		***Macro*** consisting	of ***sectors***	$\longrightarrow$	***inter-sectorial*** relations
$\cdots$	γ_N	***sectors*** γ	γ_2	γ_1	$Spec$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\uparrow$ ***intra-sectorial***
$\cdots$	$\pi_{\gamma_N}\vdots$	$\pi_{\gamma}\vdots$	$\pi_{\gamma_2}\vdots$	$\pi_{\gamma_1}\vdots$	$\Vert$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\downarrow$ ***invisible Micro***

The concept of sectors defined in this way as Micro-Macro boundaries between invisible Micro & visible Macro realizes the theoretical framework of quadrality scheme which provides the precise formulation of "quantum-classical correspondence".

A.6 Disjointness vs. Quasi-equivalence

Along this line, we clarify the homotopical basis of Tomita theorem of central decomposition of states and representations [BR].

In the C*-category $Rep_{\mathcal{X}}$ of representations of a C*-algebra $\mathcal{X}$, there exists the *universal representation* $\pi_u = (\pi_u, \mathfrak{H}_u) \in Rep_{\mathcal{X}}$ containing $\forall \pi = (\pi, \mathfrak{H}_{\pi}) \in Rep_{\mathcal{X}}$ as its subrepresentation: $\pi_u \succeq \pi = (\pi, \mathfrak{H}_{\pi}) \in Rep_{\mathcal{X}}$.

Such π_u can be concretely realized as the direct sum $(\pi_u, \mathfrak{H}_u) := \bigoplus_{\omega \in E_{\mathcal{X}}} (\pi_{\omega}, \mathfrak{H}_{\omega})$ of all the GNS representations, with the action of universal enveloping von Neumann algebra

$$\mathcal{X}'' \cong \mathcal{X}^{**} \cong \pi_u(\mathcal{X})'' \curvearrowright \mathfrak{H}_u.$$

For a representation $\pi \in Rep_{\mathcal{X}}$ its "disjoint complement" $\pi^{\overset{\shortmid}{\circ}}$ is defined [IO04a] as maximal representation disjoint from π:

$$\pi^{\overset{\shortmid}{\circ}} := \sup\{\rho \in Rep_{\mathcal{X}}; \rho \overset{\shortmid}{\circ} \pi\},$$

where $\rho \overset{\mid}{\circ} \pi \iff Rep_{\mathcal{X}}(\rho \to \pi) = \{0\}$: i.e., no non-zero intertwiners.

A.7 Disjoint Complements & Quasi-equivalence

Then, we observe the following four points, i) – v) [IO04a]:

i) $P(\pi^{\overset{\mid}{\circ}}) = c(\pi)^{\perp}$,
$P(\pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}}) = c(\pi)^{\perp\perp} = c(\pi) := \bigvee_{u \in \mathcal{U}(\pi(\mathcal{X})')} u P_\pi u^* \in \mathcal{P}(\mathfrak{Z}(W^*(\mathcal{X})))$,
where $P(\pi) \in W^*(\mathcal{X})'$ is defined as the projection corresponding to $(\pi, \mathfrak{H}_\pi)$ in $\mathfrak{H}_u$ and $c(\pi)$ is the central support of $P(\pi)$ defined by the minimal central projection majorizing $P(\pi)$ in the center $\mathfrak{Z}(W^*(\mathcal{X})) := W^*(\mathcal{X}) \cap W^*(\mathcal{X})'$ of $W^*(\mathcal{X})$.

ii) $\pi_1^{\overset{\mid}{\circ}\overset{\mid}{\circ}} = \pi_2^{\overset{\mid}{\circ}\overset{\mid}{\circ}} \iff \pi_1 \approx \pi_2$ (: quasi-equivalence= unitary equivalence up to multiplicity $\iff \pi_1(\mathcal{X})'' \simeq \pi_2(\mathcal{X})'' \iff c(\pi_1) = c(\pi_2) \iff W^*(\pi_1)_* = W^*(\pi_2)_*$)

A.8 Quasi-equivalence & Modular Structure

iii) Representation $(\pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}}, c(\pi)\mathfrak{H}_u)$ of the von Neumann algebra $W^*(\pi) \simeq \pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}}(\mathcal{X})''$ in $c(\pi)\mathfrak{H}_u = P(\pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}})\mathfrak{H}_u$ gives the ***standard form*** of $W^*(\pi)$ equipped with a normal faithful semifinite weight φ and the associated Tomita-Takesaki modular structure $(J_\varphi, \Delta_\varphi)$ [BR], whose universality is characterized by the adjunction,

$$Std(\pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}} \to \sigma) \simeq Rep_{\mathcal{X}}(\pi \to \sigma).$$

Namely, any intertwiner $T \in Rep_{\mathcal{X}}(\pi \to \sigma)$ to a standard form representation $(\sigma, \mathfrak{H}_\sigma)$ of $W^*(\sigma)$ is uniquely factored $T = T^{\overset{\mid}{\circ}\overset{\mid}{\circ}} \circ \eta_\pi$ through the canonical homotopy $\eta_\pi \in Rep_{\mathcal{X}}(\pi \to \pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}})$ with $\exists! T^{\overset{\mid}{\circ}\overset{\mid}{\circ}} \in Rep_{\mathcal{X}}(\pi^{\overset{\mid}{\circ}\overset{\mid}{\circ}} \to \sigma)$.

A.9 Symmetry and Fixed-point subalgebra

Let a physical system be described by the algebra $\mathcal{X}$ of its physical variables. Under action $\alpha = (\alpha_g)_{g \in G}$ of a Lie group G via automorphisms α_g on $\mathcal{X}$, the observable algebra $\mathcal{A}$ is defined as G-invariant subalgebra of $\mathcal{X}$ by

$$\mathcal{A} = \mathcal{X}^G := \{A; \alpha_g(A) = A \text{ for } \forall g \in G\}.$$

Under suitable assumptions, an exact sequence

$$\mathcal{A} \hookrightarrow \mathcal{X} \twoheadrightarrow \mathcal{X}/\mathcal{A} \cong \widehat{G}$$

arises in this situation, from which total algebra $\mathcal{X}$ can be recovered from the observable algebra $\mathcal{A}$ [DR89, DR90] by means of the crossed product of $\widehat{G}$ in the context of the categorical adjunction:

$$\mathcal{A} = \mathcal{X}^G \leftrightarrows \mathcal{X} = \mathcal{A} \triangleleft \widehat{G}.$$

When we combine the inclusion relation of groups controlled by the exact sequence $H \hookrightarrow G \twoheadrightarrow G/H$ with the group actions on the algebras of physical variables, we encounter the situation of symmetry breakings which involves the mutual relations among various subalgebras $\mathcal{X}^G \hookrightarrow \mathcal{X}^H \hookrightarrow \mathcal{X}$.

B Group & Representations in Categorical Context

In view of the definition for a group representation $\gamma \in Rep_G$ given by the group homomorphism properties $\gamma(g_1 g_2) = \gamma(g_1)\gamma(g_2), \gamma(e) = id_{V_\gamma}$, $\gamma(g^{-1}) = \gamma(g)^{-1}$, a G-representation γ can be viewed as a ***functor*** from the group G as a one-object category $G = \mathcal{C}_G$ consisting of an object $*$ and of group elements $g \in G$ as morphisms $* \xrightarrow{g} * \in G = Mor(\mathcal{C}_G))$ to another category $Hom(V_\gamma)$ consisting of continuous linear operators in the Hilbert space V_γ. From this categorical viewpoint, the intertwiner $T \in Rep_G(\gamma_1 \longrightarrow \gamma_2)$ from γ_1 to γ_2 is to be interpreted as a ***natural transformation*** from a functor γ_1 to another one γ_2 characterized by the commutativity diagram. In this way, the totality Rep_G of G-representations can be viewed as a category $Hilb^G$ of functors from the group G as a category $\mathcal{C}_G = G$ to the category $Hilb$ of Hilbert spaces with morphisms given by G-intertwiners as natural transformatrions. In this context, the group induction Ind_H^G from the functor category Rep_H of H-representations to that Rep_G of G-representations can be viewed as a natural transformation $Ind_H^G : Rep_H \longrightarrow Rep_G$ (preserving the tensor product structures of Rep_H and Rep_G: $Ind_H^G(\sigma_1 \otimes \sigma_2) = Ind_H^G(\sigma_1) \otimes Ind_H^G(\sigma_2)$ for $\sigma_1, \sigma_2 \in Rep_H$).

B.1 Kan Extensions as Categorical Inductions

Given a funcor $K : \mathcal{B} \to \mathcal{A}$ from a category $\mathcal{B}$ to $\mathcal{A}$ we consider the problem of extending a given functor $S : \mathcal{B} \to \mathcal{M}$ from $\mathcal{B}$ to $\mathcal{M}$ into one $T : \mathcal{A} \to \mathcal{M}$ from $\mathcal{A}$ to $\mathcal{M}$ so as to satisfy the relation $T \circ K = S$:

$$\begin{array}{ccc} & T? & \\ \mathcal{A} & \dashrightarrow & \mathcal{M} \\ K\uparrow & \circlearrowleft \quad \nearrow S & \\ \mathcal{B} & & \end{array} .$$

In this situation, the functor T is called a Kan extension [MacL] of functor S along functor K.

B.2 From Kan Extension to Induced Representation

For instance, if we identify $K : \mathcal{B} \to \mathcal{A}$ as the inclusion $\iota : H \hookrightarrow G$ of a subgroup H into the total group G and $S : \mathcal{B} \to \mathcal{M}$ as a representation $\sigma : H \to \mathcal{M} = Hilb$ of H with

$Hilb$ identified with the category of Hilbert spaces, then $T : \mathcal{A} \to \mathcal{M}$ corresponds to an extension of H-representation σ to G-representation γ:

$$\begin{array}{ccc} G & \overset{\gamma?}{\dashrightarrow} & \mathcal{M} \\ \iota \uparrow & \circlearrowleft \;\; \nearrow \sigma & \\ H & & \end{array},$$

since the commutativity $\sigma = \gamma \circ \iota$ of the diagram means $\sigma = \gamma \upharpoonright_H$. In this sense, the Kan extension can be viewed as a categorical version of the induced representations of groups.

B.3 Kan Extension and Yoneda Lemma

In view of the important roles played by natural transformations in mediating adjoint functors, we need to distinguish between the right & left Kan extensions as follows:

$$Nat_{\mathcal{M}^{\mathcal{B}}}(T \circ K \longrightarrow S) \leftrightarrows Nat_{\mathcal{M}^{\mathcal{A}}}(T \longrightarrow Ran_K(S))$$

$$Nat_{\mathcal{M}^{\mathcal{A}}}(Lan_K(S) \longrightarrow T) \leftrightarrows Nat_{\mathcal{M}^{\mathcal{B}}}(S \longrightarrow T \circ K)$$

The concept of *Yoneda embedding* [MacL]:

$$\mathbf{y}_c(-) = \mathcal{C}((-) \longrightarrow c) \in Sets^{\mathcal{C}^{op}} : \mathcal{C} \ni d \longmapsto \mathcal{C}(c \longleftarrow d) \in Sets$$

gives an embedding of a category $\mathcal{C}$ into the category $Sets^{\mathcal{C}^{op}}$ of pre-sheaves on $\mathcal{C}$ (as a categorical generalization of the concept of functions), and hence, it would be quite useful to consider the Kan extensions $Ran_{\mathbf{y}_c}$ or $Lan_{\mathbf{y}_c}$ along $K = \mathbf{y}_c$. However, systematic investigation on this topic should be done on the next occasions.

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