Note on the physical meaning of the cosmological term

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Abstract

At first glance, the issue of dressed photon in the field of nano-optics seems to have nothing to do with cosmology which deals with phenomena with the largest spatial scales in nature. However, recent preliminary analyses on the mathematical structure of Clebsch dual field introduced as a part of explaining the generating mechanism of dressed photon imply the possibility that the emergence of the cosmological constant λ as the coefficient of the cosmological term λg_{ab} may be explained by the dynamical process of simultaneous conformal symmetry breaking of electromagnetic and gravitational fields. In this short note, as a supplemental explanation of this conjecture, we give a new explanation of the physical meaning of the cosmological term λg_{ab} by proving the hitherto unnoticed identity (1) in section 1.

1 Aim of this short note

The aim is to show that the following second rank tensor ${Y_a}^b$ defined in terms of the conformal Weyl tensor W_{abcd} vanishes identically, i.e.,

$$ {Y_a}^b := W_{acde}W^{bcde} - \frac{1}{4}W_{ijkl}W^{ijkl}{g_a}^b = 0, \tag{1}$$

which provides a supplemetal proof that the cosmological term λg_{ab} represents the energy-momentum tensor of Weyl field.

2 Calculations

2.1 Preparatory process

Since Eq. (1) is a tensor equation, it suffices to show it in an inertial reference frame where the metric tensor g_{ab} are diagonalized, i.e., ($g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$) and ($g_{ab} = 0; a \neq b$). The basic properties of Weyl tensor to be used in our calculations are summarized as follows:

$$W_{abcd} = -W_{abdc}, \quad W_{abcd} = -W_{bacd}, \quad W_{abcd} = W_{cdab}, \tag{2}$$

$$W_{abcd} + W_{acdb} + W_{adbc} = 0, \quad W^{a}_{\ bca} = -W^{a}_{\ bac} = 0. \tag{3}$$

As the preparatory steps, we rewrite the last equation in (3): $W^{a}_{\ bac} = 0$ as a collection of specific forms. Since

$$W^{a}_{\ bac} = W^{0}_{\ b0c} + W^{1}_{\ b1c} + W^{2}_{\ b2c} + W^{3}_{\ b3c} = W_{0b0c} - W_{1b1c} - W_{2b2c} - W_{3b3c}, \tag{4}$$

as the Case I, when $b \neq c$, specific (aI:$1 \leq I \leq 6$) equations shown below (5) can be concisely written as

$$W_{ijik} = \hat{\epsilon} W_{ljlk}, \quad (i \neq j \neq k \neq l) \tag{5}$$

where $\hat{\epsilon} = -1$ if either j or k is zero and $\hat{\epsilon} = 1$ if neither j or k is zero.
(a1) $(b = 0, c = 1)$ and $(b = 1, c = 0)$ $\Rightarrow$ $W_{2021} + W_{3031} = 0.$
(a2) $(b = 0, c = 2)$ and $(b = 2, c = 0)$ $\Rightarrow$ $W_{1012} + W_{3032} = 0.$
(a3) $(b = 0, c = 3)$ and $(b = 3, c = 0)$ $\Rightarrow$ $W_{1013} + W_{2023} = 0.$
(a4) $(b = 1, c = 2)$ and $(b = 2, c = 1)$ $\Rightarrow$ $W_{0102} - W_{3132} = 0.$
(a5) $(b = 1, c = 3)$ and $(b = 3, c = 1)$ $\Rightarrow$ $W_{0103} - W_{2123} = 0.$
(a6) $(b = 2, c = 3)$ and $(b = 3, c = 2)$ $\Rightarrow$ $W_{0203} - W_{1213} = 0.$

As the Case II, where we have $b = c$, by introducing the extended specific (aI) equations ($7 \leq I \leq 14$) below (6), we can show that (a12), (a13) and (a14) can be concisely written as

$$W_{ijij} = -W_{klkl}, \quad (i \neq j \neq k \neq l) \tag{6}$$

(a7) $(b = 0, c = 0)$ $\Rightarrow$, $W_{1010} + W_{2020} + W_{3030} = 0.$
(a8) $(b = 1, c = 1)$ $\Rightarrow$ $W_{0101} - W_{2121} - W_{3131} = 0.$
(a9) $(b = 2, c = 2)$ $\Rightarrow$, $W_{0202} - W_{1212} - W_{3232} = 0.$

(a10) $(b=3, c=3) \quad \Rightarrow \quad W_{0303} - W_{1313} - W_{2323} = 0.$

Using (a7), (a8), (a9) and (a10), we get
(a11) $W_{2323} + W_{3131} + W_{1212} = 0,$
and combining (a11) respectively with (a8), (a9) and (a10) yield
(a12) $W_{0101} + W_{2323} = 0.$
(a13) $W_{0202} + W_{3131} = 0.$
(a14) $W_{0303} + W_{1212} = 0.$

2.2 Step 1: On off-diagonal components

In this subsection, we are going to show that $Y_{ij} = W_{icde}W_j^{cde} = 0, (i \neq j)$. Straightforward summations with respect to c, d and e, using the properties (2) yield

$$\begin{aligned}
Y_{ij}/2 &= W_{i001}W_j^{\ 001} + W_{i002}W_j^{\ 002} + W_{i003}W_j^{\ 003} \\
&+ W_{i012}W_j^{\ 012} + W_{i013}W_j^{\ 013} + W_{i023}W_j^{\ 023} \\
&+ W_{i101}W_j^{\ 101} + W_{i102}W_j^{\ 102} + W_{i103}W_j^{\ 103} \\
&+ W_{i112}W_j^{\ 112} + W_{i113}W_j^{\ 113} + W_{i123}W_j^{\ 123} \\
&+ W_{i201}W_j^{\ 201} + W_{i202}W_j^{\ 202} + W_{i203}W_j^{\ 203} \\
&+ W_{i212}W_j^{\ 212} + W_{i213}W_j^{\ 213} + W_{i223}W_j^{\ 223} \\
&+ W_{i301}W_j^{\ 301} + W_{i302}W_j^{\ 302} + W_{i303}W_j^{\ 303} \\
&+ W_{i312}W_j^{\ 312} + W_{i313}W_j^{\ 313} + W_{i323}W_j^{\ 323}. \qquad (7)
\end{aligned}$$

Rewriting contra-variant expressions, for example $W_j^{\ 001} = -W_{j001}$, (7) becomes

$$\begin{aligned}
Y_{ij}/2 &= -W_{i001}W_{j001} - W_{i002}W_{j002} - W_{i003}W_{j003} \\
&+ W_{i012}W_{j012} + W_{i013}W_{j013} + W_{i023}W_{j023} \\
&+ W_{i101}W_{j101} + W_{i102}W_{j102} + W_{i103}W_{j103} \\
&- W_{i112}W_{j112} - W_{i113}W_{j113} - W_{i123}W_{j123} \\
&+ W_{i201}W_{j201} + W_{i202}W_{j202} + W_{i203}W_{j203} \\
&- W_{i212}W_{j212} - W_{i213}W_{j213} - W_{i223}W_{j223} \\
&+ W_{i301}W_{j301} + W_{i302}W_{j302} + W_{i303}W_{j303} \\
&- W_{i312}W_{j312} - W_{i313}W_{j313} - W_{i323}W_{j323}. \qquad (8)
\end{aligned}$$

Now, based on the expression (8), let us calculate, for instance, Y_{01}. For $i = 0, j = 1$, we see that all the terms in the upper half of (8) vanish, so that we have

$$\begin{aligned} Y_{01}/2 &= W_{0201}W_{1201} + W_{0202}W_{1202} + W_{0203}W_{1203} \\ &- W_{0212}W_{1212} - W_{0213}W_{1213} - W_{0223}W_{1223} \\ &+ W_{0301}W_{1301} + W_{0302}W_{1302} + W_{0303}W_{1303} \\ &- W_{0312}W_{1312} - W_{0313}W_{1313} - W_{0323}W_{1323}. \end{aligned} \tag{9}$$

Rearranging terms in (9), we get

$$\begin{aligned} Y_{01}/2 &= (W_{0203}W_{1203} - W_{0312}W_{1312}) + (W_{0302}W_{1302} - W_{0213}W_{1213}) \\ &= (W_{0201}W_{1201} - W_{0323}W_{1323}) + (W_{0301}W_{1301} - W_{0223}W_{1223}) \\ &+ (W_{0202}W_{1202} - W_{0212}W_{1212} + W_{0303}W_{1303} - W_{0313}W_{1313}). \end{aligned} \tag{10}$$

♢ Using (a6), the first term $W_{0203}W_{1203} - W_{0312}W_{1312}$ can be rewritten as $W_{1213}(W_{1203} - W_{0312}) = 0$ and the second term $W_{0302}W_{1302} - W_{0213}W_{1213}$ can also be rewritten as $W_{1213}(W_{1302} - W_{0213}) = 0$.

♢ Succeeding applications of (a2) and (a4) to the third term respectively yield $W_{0201}W_{1201} - W_{0323}W_{1323} = W_{3032}(W_{0201} - W_{1323}) = 0$.

♢ Succeeding applications of (a3) and (a5) to the fourth term respectively yield $W_{0301}W_{1301} - W_{0223}W_{1223} = W_{1301}(W_{0301} + W_{1223}) = 0$.

♢ Using (a1), the fifth term $W_{0202}W_{1202} - W_{0212}W_{1212} + W_{0303}W_{1303} - W_{0313}W_{1313}$ becomes $W_{1202}(W_{0202} - W_{1212} - W_{0303} + W_{1313})$ which vanishes by (6).

Similarly, for Y_{23}, again from (8), we have

$$\begin{aligned} Y_{23}/2 &= -W_{2001}W_{3001} - W_{2002}W_{3002} - W_{2003}W_{3003} \\ &+ W_{2012}W_{3012} + W_{2013}W_{3013} + W_{2023}W_{3023} \\ &+ W_{2101}W_{3101} + W_{2102}W_{3102} + W_{2103}W_{3103} \\ &- W_{2112}W_{3112} - W_{2113}W_{3113} - W_{2123}W_{3123}. \end{aligned} \tag{11}$$

As in the case of Y_{01}, rearranging the terms in (11), we get

$$\begin{aligned} Y_{23}/2 &= (-W_{2001}W_{3001} - W_{2123}W_{3123}) + (W_{2023}W_{3023} + W_{2101}W_{3101}) \\ &+ W_{3012}(W_{2012} + W_{3103}) + W_{2013}(W_{3013} + W_{2102}) \\ &- [W_{3002}(W_{2002} + W_{3003}) + W_{3112}(W_{2112} + W_{3113})]. \end{aligned} \tag{12}$$

$\diamondsuit$ Succeeding applications of (a4) and (a5) to the first term respectively yield $W_{0102}(W_{3001} + W_{2123}) = 0$.

$\diamondsuit$ Succeeding applications of (a2) and (a3) to the second term respectively yield $W_{3023}(W_{2023} + W_{3101}) = 0$.

$\diamondsuit$ The third and fourth terms vanish by the use of (a1).

$\diamondsuit$ Using (a6), the fifth term can be rewritten as $-W_{0203}(W_{0202} + W_{3131} + W_{0303} + W_{1212})$ which vanishes by (6).

We can show that other off-diagonal terms also vanish by similar calculations.

2.3 Step 2: On diagonal components

Using the mixed form of (1), the diagonal components satisfy

$$W_{0cde}W^{0cde} = W_{1cde}W^{1cde} = W_{2cde}W^{2cde} = W_{3cde}W^{3cde} = \frac{1}{4}W_{ijkl}W^{ijkl}, \quad (13)$$

so that it suffices to show that

$$\hat{Y}^{(i)}_{(i)} - \hat{Y}^{(j)}_{(j)} = 0, \quad \hat{Y}^{(i)}_{(i)} := W_{(i)cde}W^{(i)cde}, \quad (14)$$

where (i) and (j) denote suffixes to which we do not apply Einstein summation convention. In what follows, as a specific example, we show that $\hat{Y}^0_0 - \hat{Y}^1_1 = 0$. Summations with respect to c and d yield

$$\begin{aligned}
\hat{Y}^0_0 &= W_{00de}W^{00de} + W_{01de}W^{01de} + W_{02de}W^{02de} + W_{03de}W^{03de} \\
&= W_{010e}W^{010e} + W_{011e}W^{011e} + W_{012e}W^{012e} + W_{013e}W^{013e} \\
&+ W_{020e}W^{020e} + W_{021e}W^{021e} + W_{022e}W^{022e} + W_{023e}W^{023e} \\
&+ W_{030e}W^{030e} + W_{031e}W^{031e} + W_{032e}W^{032e} + W_{033e}W^{033e}. \quad (15)
\end{aligned}$$

For the sake of the simplicity of notation, we introduce a symbol $(abcd) := W_{(a)(b)(c)(d)}W^{(a)(b)(c)(d)}$. Using this symbol, (15) turns out to be

$$\begin{aligned}
\hat{Y}^0_0 &= [(0100)+(0101)+(0102)+(0103)] \\
&+ [(0110)+(0111)+(0112)+(0113)] \\
&+ [(0120)+(0121)+(0122)+(0123)] \\
&+ [(0130)+(0131)+(0132)+(0133)] \\
&+ [(0200)+(0201)+(0202)+(0203)] \\
&+ [(0210)+(0211)+(0212)+(0213)] \\
&+ [(0220)+(0221)+(0222)+(0223)] \\
&+ [(0230)+(0231)+(0232)+(0233)] \\
&+ [(0300)+(0301)+(0302)+(0303)] \\
&+ [(0310)+(0311)+(0312)+(0313)] \\
&+ [(0320)+(0321)+(0322)+(0323)] \\
&+ [(0330)+(0331)+(0332)+(0333)] \\
&= 2[(0101)+(0102)+(0103)+(0112)+(0131)+(0123) \\
&+ (0201)+(0202)+(0212)+(0231)+(0223)+(0203) \\
&+ (0301)+(0302)+(0303)+(0312)+(0331)+(0323)]. \qquad (16)
\end{aligned}$$

Similar calculations for $\hat{Y}^1_1$ leads to

$$\begin{aligned}
\hat{Y}^1_1 &= 2[(0101)+(0102)+(0103)+(0112)+(0131)+(0123) \\
&+ (1201)+(1202)+(1203)+(1212)+(1231)+(1223) \\
&+ (3101)+(3102)+(3103)+(3112)+(3131)+(3123)]. \qquad (17)
\end{aligned}$$

Subtracting (17) from (16), we get

$$\begin{aligned}
(\hat{Y}^0_0 &- \hat{Y}^1_1)/2 \qquad (18) \\
&= [(0201)+(0202)+(0212)+(0231)+(0223)+(0203) \\
&+ (0301)+(0302)+(0303)+(0312)+(0331)+(0323)] \\
&- [(1201)+(1202)+(1203)+(1212)+(1231)+(1223) \\
&+ (3101)+(3102)+(3103)+(3112)+(3131)+(3123)]. \qquad (19)
\end{aligned}$$

We rearrange the terms in (19) as follows.

$$
\begin{aligned}
(\hat{Y}_0^0 &- \hat{Y}_1^1)/2 \qquad (20)\\
&= \{[(0212)-(1202)]+[(0231)-(3102)]+[(0312)-(1203)]\\
&+ [(0331)-(3103)]\}\\
&+ \{[(0203)-(1231)]+[(0302)-(3112)]\}\\
&+ \{[(0323)-(1201)]+[(0223)-(3101)]+[(0201)-(3123)]\\
&+ [(0301)-(1223)]\}\\
&+ \{[(0202)-(3131)]+[(0303)-(1212)]\}. \qquad (21)
\end{aligned}
$$

Using the last property of Weyl tensor in (2), namely, $W_{abcd}=W_{cdab}$, we see that the four terms in the first group vanish, namely, $(0212)-(1202)=0$, $(0231)-(3102)=0$, $(0312)-(1203)=0$, $(0331)-(3103)=0$. Two terms in the second group vanish by (a6) and the four terms in the third group vanish respectively by (a2), (a3), (a4) and (a5). Two terms in the last group vanish by (6). Thus, we have shown

$$\hat{Y}_0^0-\hat{Y}_1^1=0. \qquad (22)$$

Repeating similar manipulations, we can show that $\hat{Y}_0^0=\hat{Y}_2^2=\hat{Y}_3^3$.