Progresses in theoretical studies of off-shell science for dressed photons

M. Ohtsu

Research Origin for Dressed Photon,
3-13-19 Moriya-cho, Kanagawa-ku, Yokohama, Kanagawa 221-0022 Japan

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Abstract
This article reviews recent theoretical studies of off-shell science that have been launched recently based on quantum field theory, category algebra theory, quantum measurement theory, and quantum walk theory. Studies on the Majorana field have revealed a new dynamic channel that explains the enigmatic phenomena of the dressed photon (DP), and concludes that the DP is generated by pair annihilation of unstable timelike Majorana particles. Furthermore, the DP constant is defined as the third component of natural units. To describe the interacting quantum fields (including the DP) by category algebra, general relationships among spacelike events are studied to deal with ordered sets (causality structures) and groups (symmetry structures) as relativity in a generalized sense. Relativity and the quantum nature are integrated as a category theoretical structure and as a noncommutative probabilistic structure, respectively. Measurement theory is reviewed for describing the interacting quantum field by using algebraic quantum field theory based on a local net. Finally, quantum walk models are reviewed for numerically analyzing the unique nature of DP energy transfer. After defining the Grover walk and the boundary operators, the centered generalized eigenspace is characterized by using the concept of flow from graph theory. Studies on counter-intuitive phenomena for the Grover walk on a general connected graph by using spectral analysis are also reviewed.

1. Introduction

On-shell science, which is the established basis of conventional optical science, has never succeeded in theoretically analyzing a variety of unique optical phenomena found through experimental studies on the dressed photon (DP) [1-5]. This is because on-shell science has never dealt with light–matter interactions in a nanometer-sized space, despite such interactions being indispensable for generating the DP. Fortunately, studies on off-shell science (the complement of on-shell science) have been launched recently to deal with this interaction and have produced several significant outcomes [6-9].

This article reviews recent progress in theoretical studies of off-shell science that were conducted for analyzing the unique phenomena originating from the DP. It summarizes the topics discussed in the original papers that were recently published in the Special Issue "Quantum Fields and Off-Shell Science" of the academic journal *Symmetry*, for which the author served as the Guest Editor [8,10-16].

Noting that the DP is a quantum field generated in a complex system composed of photons and electrons (or excitons) in a nanometer-sized material, Section 2 reviews physical and mathematical bases for describing a DP as an off-shell quantum field. Section 3 discusses how to describe the quantum fields, including the DP, by category algebra. Section 4 introduces recent studies on measurement theory for off-shell quantum fields. Section 5 is devoted to quantum walk models for describing unique phenomena of DP energy transfer and compares the models' predictions with experimental results. This article concludes with a summary in Section 6.

2. Describing the dressed photon as an off-shell quantum field

The DP field has been recently formulated by using the Clebsch dual variable, motivated by fluid dynamics [17-19]: The Clebsch parametrization of the rotational model of the velocity field U_μ was formulated of the form $U_\mu = \lambda \nabla_\mu \phi$ with two scalar fields, λ and ϕ. By defining the covariant vectors $C_\mu = \nabla_\mu \phi$ and $L_\mu = \nabla_\mu \lambda$ and the bi-vector $S_{\mu\nu} = C_\mu L_\nu - L_\mu C_\nu$, the energy–momentum tensor was defined by $\hat{T}_\mu^\nu = -S_{\mu\sigma} S^{\nu\sigma}$.

A simple computation [17] showed that this tensor satisfied the equation $\hat{T}_\mu^\nu = \rho C_\mu C^\nu$. By noting that this equation indicated the Veronese embedding in projective geometry, a model in an arbitrary number of dimensions was constructed, and the symmetry of the model was inspected [10]. As a result, it was found that the symmetry was described in terms of a compact homogeneous space, such as Grassmann manifolds and flag manifolds, as well as pre-homogeneous vector spaces. It should be noted that the discussion above was not restricted to a specific number of dimension, so that it could be used to formulate in an arbitrary number of dimensions. By noting so, after surveying the complex number field $\boldsymbol{C}$ in an arbitrary number of dimensions $n \geq r \geq s$, the DP field with $n = 4$, $r = 2$, and $s = 1$ in the real number field $\boldsymbol{R}$ was studied.

The symmetry of equations for the DP could be described in a general manner: The tensor S was understood as an affine version of Plücker coordinates of the Grassmann manifold *Grass*(4, 2, R). The splitting expression of the tensor $\hat{T}$ was related with an affine version of the flag manifold *Flag*(4; 1, 2, R). Furthermore, the special fiber of the homogeneous bundle could be chosen by setting the off-shell conditions [18] $C_\nu C^\nu = 0$, $L_\nu C^\mu = 0$, and $L_\nu L^\mu = -\rho$.

With the help of the mathematical discussions on the symmetry of the DP above, intensive studies on the essential role played by the spacelike four-momentum in electromagnetic field interactions found that the spacelike momentum field was embodied by the Majorana fermion [7]. These studies opened up a new dynamic channel that explained the enigmatic phenomena of the DP, dark energy, and dark matter: The DP was generated by pair annihilation of unstable timelike Majorana particles. The dark energy came into existence as the compound ground state of the Majorana field. The dark matter originated from a revised cosmological term through simultaneous conformal symmetry breaking in electromagnetic and gravitational fields.

Further studies on a couple of intriguing subjects were carried out based on the new notion of simultaneous conformal symmetry breaking [8]: One was for defining the DP constant. By considering it as the third component of natural units in addition to the Planck constant and the speed of light, this constant was defined as the geometric mean of the smallest and the largest lengths (Planck length and the length relating to the cosmological constant, respectively). Interestingly, this mean value (=50 – 70 nanometers) suggested a measure of the Heisenberg cut for electromagnetic phenomena. The other was a new perspective on cosmology that combined two original notions, i.e., twin universes and conformal cyclic cosmology, proposed by Petit and Penrose, respectively, into one novel picture of a self-similarly expanding universe.

3. Describing off-shell quantum fields with category algebra

In order to describe the unique natures of the DP field (i.e., the interacting quantum field) with off-shell science [7,8], the concepts of "physical quantities" and "physical states" have to be appropriately formulated. Although quantum field theory has been constructed by the unification of relativity theory and quantum theory, it has never dealt with a non-trivial interacting quantum field theory on a four-dimensional Minkowski spacetime.

As a preliminary study for dealing with such a non-trivial theory, the concepts of category algebras were studied by building a new bridge between a generalized probability theory (known as noncommutative probability or quantum probability) and a category theory [11]. This study was carried out for forming a mathematical framework of quantum field theory in terms of states as linear functional on category algebra. Through this study, it was made clear that category algebra could be considered as generalized matrix algebra. Furthermore, the notion of state on category (i.e., linear functional on category algebra) could be considered as a conceptual generalization of probability measures on sets (i.e., discrete categories). Moreover, a representation of category algebras of †-categories on certain generalized Hilbert spaces was obtained by generalizing the famous GNS (Gelfand–Naimark–Segal) construction.

One more preliminary study was a proposal of a new axiomatic approach to nonstandard analysis and its application to the general theory of spatial structures in terms of category theory [12]. This proposal was based on the idea of internal set theory by using an endofunctor U on a topos of sets S together with a natural transformation, instead of using the terms "standard", "internal", or "external". Moreover, general notions of a space (U-space) and the category *USpace* were proposed. Their objects and morphisms were U-spaces and functions (called U-spatial morphisms), respectively. The category *USpace*, which was confirmed to be Cartesian closed, provided a unified viewpoint toward topological and coarse geometric structure. Finally, it was confirmed that the category *USpace* was useful to study symmetries/asymmetries of the quantum field (including the DP field) with infinite degrees of freedom.

With the help of the preliminary studies above, intensive studies were carried out for building a non-trivial interacting quantum field model on a four-dimensional Minkowski spacetime [13]. Their unique strategies were to investigate quantum fields in terms of category algebra, which was noncommutative over a *rig* (ring without "negatives"; an algebraic system equipped with addition and multiplication). The category and *rig* corresponded to the relativity aspect and the quantum aspect, respectively.

For describing the interacting quantum fields, general relationships among spacelike events were studied based on the concept of categories that treated ordered sets (causality structures) and groups (symmetry structures) as relativity. By utilizing category algebras and states on categories, relativity and quantum natures were integrated by considering them as a category theoretical structure and as a noncommutative probabilistic structure, respectively. Furthermore, a basic relationship was found among the category algebra, the algebraic quantum field theory [20,21], and the topological quantum field theory [22,23].

The series of studies reviewed above provided a new basis for generalizing the DHR (Doplicher–Haag–Roberts)–DR (Doplicher–Roberts) sector theory [24-30] as well as for developing the concepts of Ojima's micro–macro duality [31,32] and quadrality scheme [33] from the viewpoint of category algebras and states on categories.

4. Constructing a measurement theory for off-shell quantum fields

For constructing a DP measurement theory for off-shell fields (including the DP field), intensive studies have been carried out recently based on algebraic quantum field theory, quantum measurement theory, and their mathematics [14]. Observables of the interacting quantum fields were given by self-adjoint elements of C^*-algebras. Furthermore, a completely positive instrument and a measuring process were introduced [34]. This instrument was used for quantum mechanical modeling of measurement. In these studies, a completely positive instrument in a quantum system with "finite" degrees of freedom was defined by a measuring process.

For the quantum systems with "infinite" degrees of freedom, the theory of completely positive instruments was also developed recently in order to formulate these instruments using general von Neumann algebra [35,36]. As a result, the current measurement theory made it possible to select the components of the probability distributions and states that appeared to the macroscopic space through the measurement process.

In order to formulate the measurement theory for quantum systems described by C^*-algebras, studies on a more general case, compared to von Neumann algebras, have been required. To meet this requirement, the instruments were defined by using central subspaces of the dual of a C^*-algebra, and its consistency with the definition in the von Neumann algebraic setting was confirmed.

Furthermore, a unification of the measurement theory and the sector theory was proposed by defining and characterizing the centrality of instruments. The operational characterization and macroscopic nature of quantum measurement were also analyzed based on the disjointness of states. The results of this analysis were applied to study the systems described by C^*-algebras that were generated from field operators. They made it possible to analyze the macroscopic aspects of quantum fields in terms of measurement theory.

In the setting of algebraic quantum field theory, a local net {A(O)}O∈R1 on a space M1 was used in order to describe the DP phenomena. Toward the future, it should be noted that, in describing the measurement of the DP, the use of the local net alone is not enough. In fact, to evaluate the effect of the DP, an operation is required wherein some probe is brought closer to the nanometer-sized space in which the DP is generated.

The series of discussions above are advantageous in that the identification of sectors by the measurement is justified by the measurement-theoretic description. It is expected that the establishment of the measurement theory in quantum systems described by C*-algebras will open up new perspectives to understand macroscopic aspects of microscopic quantum systems.

5. Quantum walk model for describing unique phenomena of dressed photon energy transfer

A quantum walk (QW) model is expected to be a powerful theoretical tool for analyzing experimentally observed unique phenomena of the DP energy transfer [15,37]. As an example of such phenomena, a unique DP energy transfer [38] via nano-particles (NPs), dispersed three-dimensionally on a substrate material, has been experimentally found [39]: In spite of the fact that the number of routes to the output port increased by increasing the thickness of the dispersed NP layer, the signal intensity measured at the output port increased unexpectedly.

It is surprising to find that the theoretical studies on the QW have independently found a phenomenon that corresponds to the experimentally confirmed one described above. That is, for a three-state QW with the Grover coin [40], the survival probability on a finite line was non-vanishing [41] due to the existence of trapped states. They were the eigenstates of the unitary evolution operator that did not have a support on the sinks. Trapped states crucially affected the efficiency of quantum transport [42] and led to counter-intuitive effects; e.g., the transport efficiency could be increased by increasing the distance between the initial vertex and the sink [43,44]. This effect corresponds to the unexpectedly found experimental phenomenon above.

In order to investigate this correspondence, counter-intuitive phenomena for the Grover walk on a general connected graph were studied by using spectral analysis [15]. Here, it should be noted that the Grover walk was an induced QW of the random walk, as has been proved by the spectral mapping theorem [45]. By noting so, the Grover walk with sinks was connected to the Grover walk with tails. The tails were assumed as semi-infinite paths attached to a finite and connected graph. The set of vertices connecting to the tails was called the boundary. The Grover walk with tails was introduced in terms of scattering theory [46,47]. By setting some appropriate bounded initial state so that the support was included in the tails, the existence of the fixed point of the dynamical system, induced by the Grover walk with tails, was shown. Then, the stable generalized eigenspace *Hs* , in which the dynamical system exists, was confirmed to be orthogonal to the centered generalized eigenspace *Hc* [48] at every time step [49]. The centered generalized eigenspace was generated by the generalized eigenvectors of the principal submatrix of the time evolution operator of the Grover walk with respect to the internal graph. As a result, all the corresponding absolute values of the eigenvalues were found to be 1. This eigenstate was equivalent to the attractor space [42] of the Grover walk with sinks. Indeed, it was shown that the stationary state of the Grover walk with sinks was attracted to this centered generalized eigenstate.

Furthermore, this centered generalized eigenspace was characterized by using the persistent eigenspace of the underlying random walk whose supports did not overlap with the boundary. The concept of "flow" from graph theory was also used for this characterization. As a result, it was found that the existence of the persistent eigenspace of the underlying random walk significantly influenced the asymptotic behavior of the corresponding Grover walk in spite of the fact that it had little effect on the asymptotic behavior of the random walk. It was also made clear that the graph structure constructed the symmetric or anti-symmetric flow satisfying Kirchhoff's law and contributed to the non-zero survival probability of the Grover walk, as was suggested in [42,45].

For numerically analyzing the experimentally confirmed unique DP energy transfer, practical two-dimensional and three-dimensional QW models were constructed and their spatio-temporal evolution equations were derived [37]. Furthermore, discontinuity of the energy of a QW in impurities was theoretically confirmed as a function of the inflow frequency [16].

6. Summary

This article reviewed theoretical studies of off-shell science that have been launched recently based on quantum field theory, category algebra theory, quantum measurement theory, and quantum walk theory. First, physical and mathematical bases were reviewed to consider the DP as an off-shell quantum field. Studies on the Majorana field revealed a new dynamic channel that explained the enigmatic phenomena of the DP. It was concluded that the DP was generated by pair annihilation of unstable timelike Majorana particles. Furthermore, the DP constant was defined as the third component of natural units. Second, category algebra was introduced to describe the interacting quantum fields (including the DP). For this description, general relationships among spacelike events were reviewed to deal with ordered sets (causality structures) and groups (symmetry structures) as relativity in a generalized sense. By utilizing category algebras and states on categories, relativity and quantum natures were integrated as a category theoretical structure and as a noncommutative probabilistic structure, respectively. Third, recent studies on measurement theory were reviewed for describing the interacting quantum field by using algebraic quantum field theory based on a local net. Finally, quantum walk models were reviewed for numerically analyzing the unique natures of DP energy transfer. After defining the Grover walk and the boundary operators, the centered generalized eigenspace was characterized by using the concept of flow from graph theory. Studies on counter-intuitive phenomena for the Grover walk on a general connected graph by using spectral analysis were also reviewed.

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