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Symmetry of Dressed Photon

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Abstract: Motivated by describing the symmetry of a theoretical model of dressed photons, we introduce several spaces with Lie group actions and the morphisms between them depending on three integer parameters $n \geq r \geq s$ on dimensions. We discuss the symmetry on these spaces using classical invariant theory, orbit decomposition of prehomogeneous vector spaces, and compact reductive homogeneous space such as Grassmann manifold and flag variety. Finally, we go back to the original dressed photon with $n = 4, r = 2, s = 1$.

Keywords: dressed photon; Grassmann manifold; flag manifold; pre-homogeneous vector space; invariants

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1. Introduction

A formulation of dressed photons in quantum field theory is given by the Clebsch dual variable, motivated by fluid dynamics [1–3]. The Clebsch parametrization of the rotational model of the velocity field U_μ is formulated of the form $U_\mu = \lambda \nabla_\mu \phi$ with two scalar fields λ, ϕ. We define the covariant vectors $C_\mu = \nabla_\mu \phi$ and $L_\mu = \nabla_\mu \lambda$, and the bi-vector $S_{\mu\nu} = C_\mu L_\nu - L_\mu C_\nu$. The energy–momentum tensor is defined by $\hat{T}^\nu_\mu = -S_{\mu\sigma} S^{\nu\sigma}$. It is shown

$$\hat{T}^\nu_\mu = \rho C_\mu C^\nu \tag{1}$$

by a simple computation [1].

Our main concern is this last Equation (1). This looks like Veronese embedding in projective geometry. In this paper, we introduce the model in arbitrary dimension and describe the symmetry of this model. Most of the material comes from the modern treatment of classical invariant theory [4,5]. Especially, the quadratic map arising in reductive dual pair [6,7] is used as one of the key ingredients in this paper to construct geometric objects describing the symmetry. This enables us to give another explanation of the last Equation (1) on $\hat{T}$.

Physical study of dressed photons, including experiments and related applications, called dressed photon phenomenon, has already been summarized in our previous paper [8]. This paper serves as a complementary observation on symmetry of theoretical foundations of dressed photon Equation [1], which would be expected as is in classical electromagnetism. We conclude that the symmetry is well described in terms of compact homogeneous space, such as Grassmann manifolds and flag manifolds, as well as pre-homogeneous vector spaces, which is not a homogeneous space, but still has a large symmetry. It is also significant that a part of discussion is not restricted to a specific dimension, so that half of them are formulated in arbitrary dimension.

The construction of this paper is as follows: In Section 2, we work over the complex number field $\mathbb{C}$, and do si in arbitrary dimensions $n \geq r \geq s$. In Section 3, we consider the special case $n = 4, r = 2, s = 1$ with the real number field $\mathbb{R}$. The symmetry and invariants are mostly the same for $\mathbb{C}$ and for $\mathbb{R}$; however, there is a subtle and rather complicated problem on connected components over $\mathbb{R}$. In order to concentrate this complication for $\mathbb{R}$, the common features of the model are discussed over $\mathbb{C}$, and the different point is separately treated in Section 3.

2. The Model over the Complex Numbers

2.1. Symmetry in Arbitrary Dimension

Let $M(n,r,\mathbb{C})$, $\mathrm{Sym}(n,\mathbb{C})$, $\mathrm{Alt}(n,\mathbb{C})$ be the set of n by r matrices, symmetric matrices, and skew-symmetric matrices with complex entries. We denote by $M(n,r,\mathbb{C})_{\mathrm{rk}\le i}$ the subset consisting of matrices of rank at most i. The transpose of a matrix X is denoted by X^T. Classical invariant theory gives the following maps:

Let $J \in \mathrm{Alt}(r,\mathbb{C})_{\mathrm{rk}=r}$. We define the map

$$\mathbb{S} : M(n,r,\mathbb{C}) \longrightarrow \mathrm{Alt}(n,\mathbb{C}) \quad \text{by} \quad X \mapsto XJX^T.$$

If $r \ge n$, then this map is surjective. If $r < n$, then the image of this map is $\mathrm{Alt}(n,\mathbb{C})_{\mathrm{rk}\le r}$. This map is $GL(n,\mathbb{C}) \times Sp(r,\mathbb{C})$-equivariant, in the sense that $\mathbb{S}(lXh) = l\,\mathbb{S}(X)\,l^T$ for any $l \in GL(n,\mathbb{C})$ and $h \in Sp(r,\mathbb{C})$, where the symplectic group attached to J is defined by $Sp(r,\mathbb{C}) = Sp(J,\mathbb{C}) = \{h \in M(r,\mathbb{C}) \mid hJh^T = J\}$.

Let $g \in \mathrm{Sym}(n,\mathbb{C})_{\mathrm{rk}=n}$. We define the map

$$\mathbb{G} : M(n,r,\mathbb{C}) \longrightarrow \mathrm{Sym}(r,\mathbb{C}) \quad \text{by} \quad X \mapsto X^T g X.$$

If $r \ge n$, then this map is surjective. If $r < n$, then the image of this map is $\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le n}$. This map is $O(n,\mathbb{C}) \times GL(r,\mathbb{C})$-equivariant, in the sense that $\mathbb{G}(lXh) = h^T\mathbb{G}(X)h$ for any $l \in O(n,\mathbb{C})$ and $h \in GL(r,\mathbb{C})$, where the orthogonal group attached to g is defined by $O(n,\mathbb{C}) = O(g,\mathbb{C}) = \{l \in M(n,\mathbb{C}) \mid l^T g l = g\}$. Especially, put $r = n$ and restrict the domain, we define

$$\mathbb{T} : \mathrm{Alt}(n,\mathbb{C}) \longrightarrow \mathrm{Sym}(n,\mathbb{C}) \quad \text{by} \quad X \mapsto XgX^T = -XgX = X^T g X.$$

This is $O(n,\mathbb{C})$-equivariant: $\mathbb{T}(lXl^T) = l\,\mathbb{T}(X)\,l^T$.

From now on, we assume that $n \ge r \ge s$. Each $GL(r,\mathbb{C})$-orbit on $\mathrm{Sym}(r,\mathbb{C})$ is parametrized by the rank. The closure relation of orbits is linear, so that the closure of $\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}=s}$ is $\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s}$. We define

$$Y(\mathbb{C}) = M(n,r,\mathbb{C})_{\mathrm{rk}=r} \cap \mathbb{G}^{-1}(\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s})$$

Our main target is the description of the image of $Y(\mathbb{C})$ by the map $\mathbb{T} \circ \mathbb{S}$:

$$\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s} \xleftarrow{\mathbb{G}} M(n,r,\mathbb{C})_{\mathrm{rk}=r} \xrightarrow{\mathbb{S}} \mathrm{Alt}(n,\mathbb{C}) \xrightarrow{\mathbb{T}} \mathrm{Sym}(n,\mathbb{C}). \tag{2}$$

In order to state the main result, we introduce several auxiliary spaces and maps. We fix $g' \in \mathrm{Sym}(s,\mathbb{C})_{\mathrm{rk}=s}$. We define the maps

$$\mathbb{V} : M(n,s,\mathbb{C}) \longrightarrow \mathrm{Sym}(n,\mathbb{C}) \quad \text{by} \quad \mathbb{V}(X) = Xg'X^T.$$

$$\mathbb{V}' : M(r,s,\mathbb{C}) \longrightarrow \mathrm{Sym}(r,\mathbb{C}) \quad \text{by} \quad \mathbb{V}'(X') = X'g'X'^T,$$

Note that these maps are similar to $\mathbb{G}$, but transposed. Especially, the orthogonal group $O(g',\mathbb{C})$ acts transitively on each fiber of an element of $\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}=s}$.

We define $Z(\mathbb{C})$ to be the fiber product of the map $\mathbb{G} : Y(\mathbb{C}) \to \mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s}$ and $\mathbb{V}' : M(r,s,\mathbb{C})_{\mathrm{rk}=s} \to \mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s}$:

$$\begin{aligned} Z(\mathbb{C}) &= Y(\mathbb{C}) \times_{\mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s}} M(r,s,\mathbb{C})_{\mathrm{rk}=s} \\ &= \{(X,X') \in M(n,r,\mathbb{C}) \times M(r,s,\mathbb{C}) \mid \mathrm{rk}(X) = r, \mathrm{rk}(X') = s, X^T g X = X'g'X'^T\}. \end{aligned}$$

We have the commutative diagram

$$\begin{array}{ccccc} M(n,r,\mathbb{C}) & \longleftarrow & Y(\mathbb{C}) & \xleftarrow{\tilde{\mathbb{V}}'} & Z(\mathbb{C}) \\ \mathbb{G}\downarrow & & \mathbb{G}\downarrow & \square & \tilde{\mathbb{G}}\downarrow \\ \mathrm{Sym}(r,\mathbb{C}) & \longleftarrow & \mathrm{Sym}(r,\mathbb{C})_{\mathrm{rk}\le s} & \xleftarrow{\mathbb{V}'} & M(r,s,\mathbb{C})_{\mathrm{rk}=s} \end{array} \tag{3}$$

where the right square is Cartesian.

The map $\mathbb{T}\circ\mathbb{S}$ does not factor through the map $\mathbb{V}$. However, when we lift the map from $Y(\mathbb{C})$ to $Z(\mathbb{C})$, the map factor through $\mathbb{V}$. To be more precise, we have the following:

Theorem 1. $(\mathbb{T}\circ\mathbb{S})(X) = (\mathbb{V}\circ\phi)(X,X')$ *for all* $(X,X') \in Z(\mathbb{C})$, *where we define*

$$\phi : M(n,r,\mathbb{C})_{\mathrm{rk}=r} \times M(r,s,\mathbb{C})_{\mathrm{rk}=s} \longrightarrow M(n,s,\mathbb{C})_{\mathrm{rk}=s} \quad \textit{by} \quad (X,X') \mapsto XJX' \tag{4}$$

Proof. $(\mathbb{T}\circ\mathbb{S})(X) = \mathbb{T}(XJX^T) = (XJX^T)g(XJX^T)^T = XJ\mathbb{G}(X)J^TX^T$
$= XJ\mathbb{V}'(X')J^TX^T = (XJX')g'(XJX')^T = \mathbb{V}(XJX') = (\mathbb{V}\circ\phi)(X,X')$. □

This theorem is illustrated as the following commutative diagram:

$$\begin{array}{ccccccc} Z(\mathbb{C}) & \longrightarrow & M(n,r,\mathbb{C})_{\mathrm{rk}=r} \times M(r,s,\mathbb{C})_{\mathrm{rk}=s} & \xrightarrow{\phi} & M(n,s,\mathbb{C})_{\mathrm{rk}=s} \\ \tilde{\mathbb{V}}'\downarrow & & \downarrow & & \downarrow\mathbb{V} \\ Y(\mathbb{C}) & \longrightarrow & M(n,r,\mathbb{C})_{\mathrm{rk}=r} \xrightarrow{\mathbb{S}} \mathrm{Alt}(n,\mathbb{C}) & \xrightarrow{\mathbb{T}} & \mathrm{Sym}(n,\mathbb{C}) \end{array} \tag{5}$$

Note that the maps $\mathbb{S},\mathbb{G},\mathbb{T},\mathbb{V},\mathbb{V}'$ are common in classical invariant theory and theory of reductive dual pair, though the space $Y(\mathbb{C})$ and $Z(\mathbb{C})$ is unique in our setting.

2.2. Grassmann and Flag Manifold

We will show that the map ϕ introduced in Theorem 1 has an interpretation in the projective setting. We still assume $n \ge r \ge s$. The Grassmann manifold $\mathrm{Grass}(n,r,\mathbb{C})$ is the set of r-dimensional subspace of $\mathbb{C}^n$. This is identified with

$$M(n,r,\mathbb{C})_{\mathrm{rk}=r}/GL(r,\mathbb{C}) \cong \mathrm{Grass}(n,r,\mathbb{C}).$$

Every r-dimensional subspace of $\mathbb{C}^n$ is spanned by r linear independent column vectors in $\mathbb{C}^n$.

The flag manifold $\mathrm{Flag}(n;k_1,\ldots,k_m,\mathbb{C})$ is the set of flags of type $(k_1,k_2,\ldots,k_m)$, which is defined to be a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_m$ of $\mathbb{C}^n$, where $1 \le k_1 < k_2 < \cdots < k_m < n$, with $\dim V_i = k_i$ $(i = 1,\ldots,m)$. Grassmann manifold is a special case of flag manifolds with $m = 1$. On the other hand, a flag variety is regarded as the incidence variety of the product of Grassmann manifolds. For example, $\mathrm{Flag}(n;k_1,k_2,\mathbb{C}) = \{(V_1,V_2) \in \mathrm{Grass}(N,k_1,\mathbb{C}) \times \mathrm{Grass}(N,k_2,\mathbb{C}) \mid V_1 \subset V_2\}$. We have an isomorphism

$$(M(n,r,\mathbb{C})_{\mathrm{rk}=r} \times M(r,s,\mathbb{C})_{\mathrm{rk}=s})/(GL(r,\mathbb{C}) \times GL(s,\mathbb{C})) \cong \mathrm{Flag}(n;s,r,\mathbb{C}).$$

In the following commutative diagram, each space in the upper line, which arises in Theorem 1, is a locally closed subset of an affine space, while each space in the lower line is a projective variety.

$$\begin{array}{ccccc} M(n,r,\mathbb{C})_{\mathrm{rk}=r} & \longleftarrow & M(n,r,\mathbb{C})_{\mathrm{rk}=r} \times M(r,s,\mathbb{C})_{\mathrm{rk}=s} & \xrightarrow{\phi} & M(n,s,\mathbb{C})_{\mathrm{rk}=s} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Grass}(n,r,\mathbb{C}) & \longleftarrow & \mathrm{Flag}(n;s,r,\mathbb{C}) & \longrightarrow & \mathrm{Grass}(n,s,\mathbb{C}). \end{array} \tag{6}$$

The maps in the lower line are given by $V_2 \leftarrow (V_1, V_2) \mapsto V_1$. This double fibration is often used in Radon transform and Heck correspondence [9].

In the case $r = 2$, the map

$$\mathbb{S} : M(n, 2, \mathbb{C})_{\mathrm{rk}=2} \longrightarrow \mathrm{Alt}(n, \mathbb{C})_{\mathrm{rk}=2}$$

induces the Plücker embedding

$$\mathrm{Grass}(n, 2, \mathbb{C}) \longrightarrow \mathrm{Alt}(n, \mathbb{C})_{\mathrm{rk}=2}/\mathbb{C}^\times \subset \mathbb{P}^{n(n-1)/2-1}(\mathbb{C}).$$

3. The Model over Real Numbers

We now consider the special case $n = 4, r = 2, s = 1$, and replace $\mathbb{C}$ by $\mathbb{R}$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be the standard non-degenerate skew-symmetric matrix. Note that $J^T = -J$ and $\det J = 1$. Let g be the diagonal matrix with diagonal entries $(1, -1, -1, -1)$. Finally, we put $g' = 1$.

Most of the story in the previous section does hold over the real number field $\mathbb{R}$ as well. However, the disconnectedness makes things complicated. For example, although the map $\mathbb{V}' : M(2, 1, \mathbb{C}) \longrightarrow \mathrm{Sym}(2, \mathbb{C})_{\mathrm{rk}\leq 1}$ given by $\mathbb{V}'(X') = X'X'^T$ is surjective, the map $\mathbb{V}' : M(2, 1, \mathbb{R}) \longrightarrow \mathrm{Sym}(2, \mathbb{R})_{\mathrm{rk}\leq 1}$ is not surjective, because $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is not in the image. In order to improve this defect, we introduce a non-zero scalar multiplication so that we modify the map $\mathbb{V}'$ by $\mathbb{V}_2$ given below (7).

3.1. Quadratic Polynomial

Let us consider the matrix $X = (C, L) = \begin{pmatrix} C_0 & L_0 \\ C_1 & L_1 \\ C_2 & L_2 \\ C_3 & L_3 \end{pmatrix} \in M(4, 2, \mathbb{R})$ with the column vectors $C, L \in \mathbb{R}^4$. Here, $M(m, n, \mathbb{R})$ the set of m by n matrices with real coefficients. The entry of the map

$$\mathbb{S} : M(4, 2, \mathbb{R}) \ni X \mapsto XJX^T \in \mathrm{Alt}(4, \mathbb{R})_{\mathrm{rk}\leq 2}$$

is given by

$$\mathbb{S}_{\mu\nu}(X) = (XJX^T)_{\mu\nu} = C_\mu L_\nu - L_\mu C_\nu,$$

which realizes the definition of $S_{\mu\nu}$. The map $\mathbb{S}$ is $GL(4, \mathbb{R}) \times SL(2, \mathbb{R})$-equivariant, where we remark the accidental isomorphism of lower rank groups:

$$SL(2, \mathbb{R}) = \{h \in M(2, \mathbb{R}) \mid \det h = 1\} = Sp(2, \mathbb{R}) = \{h \in M(2, \mathbb{R}) \mid hJh^T = J\}$$

The action of $GL(4, \mathbb{R})$ on $\mathrm{Alt}(4, \mathbb{R})$ is prehomogeneous [10]. The image $\mathrm{Alt}(4, \mathbb{R})_{\mathrm{rk}\leq 2}$ is the complement of the open $GL(4, \mathbb{R})$-orbit $\mathrm{Alt}(4, \mathbb{R})_{\mathrm{rk}=4}$, and its defining equation is given by the basic relative invariant, Pfaffian

$$\mathrm{Pf}(S) = S_{01}S_{23} + S_{02}S_{31} + S_{03}S_{12}.$$

Then, the singular set $\mathrm{Alt}(4, \mathbb{R})_{\mathrm{rk}\leq 2} = \{S \in \mathrm{Alt}(4, \mathbb{R}) \mid \mathrm{Pf}(S) = 0\}$ is the zero locus of Pfaffian, and the open orbit $\mathrm{Alt}(4, \mathbb{R})_{\mathrm{rk}=4}$ has two connected components $\{S \in \mathrm{Alt}(4, \mathbb{R}) \mid \pm\mathrm{Pf}(S) > 0\}$. The relation $\mathrm{Pf}(S) = 0$ is considered as a Plücker relation of Grassmann manifold $\mathrm{Grass}(4, 2, \mathbb{R})$.

3.2. Symmetry Breaking

We restrict the general linear group $GL(4,\mathbb{R})$ to the subgrouop $O(1,3)$. Let $g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ be the standard non-degenerate symmetric matrix with signature $(1,3)$. Define Lorentz group (indefinite orthogonal group of signature $(1,3)$) by

$$O(1,3) = \{l \in M(4,\mathbb{R}) \mid l^T g l = g\}.$$

Gram matrix with respect to this metric is given by the map

$$\mathbb{G} : M(4,2,\mathbb{R}) \ni X \mapsto X^T g X \in \mathrm{Sym}(2,\mathbb{R})$$

where $\mathrm{Sym}(n,\mathbb{R})$ is the set of real symmetric matrices of size n. The map $\mathbb{G}$ is $O(1,3) \times GL(2,\mathbb{R})$-equivariant:

$$\mathbb{G}(lXh) = h^T \mathbb{G}(X) h, \quad \forall l \in O(1,3), h \in GL(2,\mathbb{R}).$$

We define

$$Y(\mathbb{R}) := M(4,2,\mathbb{R})_{\mathrm{rk}=2} \cap \mathbb{G}^{-1}(\mathrm{Sym}(2,\mathbb{R})_{\mathrm{rk}\leq 1}),$$

an $O(1,3) \times SL(2,\mathbb{R})$-invariant subset of $M(4,2,\mathbb{R})$. Moreover, let

$$S^1 := \{\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2 \mid v_1^2 + v_2^2 = 1\}$$

and an analogue of Veronese map is defined by

$$\mathbb{V}_2 : S^1 \times \mathbb{R}^\times \ni (\mathbf{v}, -\rho) \mapsto -\rho \mathbf{v}\mathbf{v}^T \in \mathrm{Sym}(2,\mathbb{R})_{\mathrm{rk}\leq 1}. \tag{7}$$

The fiber product of two maps

$$\begin{aligned} &\mathbb{G} : Y(\mathbb{R}) \longrightarrow \mathrm{Sym}(2,\mathbb{R})_{\mathrm{rk}\leq 1}, && \tilde{C} \mapsto \mathbb{G}(\tilde{C}), \\ &\mathbb{V}_2 : S^1 \times \mathbb{R}^\times \to \mathrm{Sym}(2,\mathbb{R})_{\mathrm{rk}\leq 1}, && (\mathbf{v}, -\rho) \mapsto -\rho \mathbf{v}\mathbf{v}^T \end{aligned}$$

is defined by

$$\begin{aligned} Z(\mathbb{R}) &:= Y(\mathbb{R}) \times_{\mathrm{Sym}(2,\mathbb{R})_{\mathrm{rk}\leq 1}} (S^1 \times \mathbb{R}^\times) \\ &= \{(X, \mathbf{v}, -\rho) \in M(4,2,\mathbb{R})_{\mathrm{rk}=2} \times S^1 \times \mathbb{R}^\times \mid \mathbb{G}(X) = -\rho \mathbf{v}\mathbf{v}^T\}, \end{aligned}$$

then we obtain a real counterpart of (3):

$$\begin{array}{ccc} Z(\mathbb{R}) & \xrightarrow{\tilde{\mathbb{V}}_2} & Y(\mathbb{R}) \\ \downarrow \tilde{\mathbb{G}} & \square & \downarrow \mathbb{G} \\ S^1 \times \mathbb{R}^\times & \xrightarrow[\mathbb{V}_2]{} & \mathrm{Sym}(2,\mathbb{R})_{\mathrm{rk}\leq 1} \end{array}$$

3.3. Tensor $\hat{T}$

The map

$$\mathbb{T} : \mathrm{Alt}(4,\mathbb{R}) \ni S \mapsto -SgS \in \mathrm{Sym}(4,\mathbb{R})$$

has been defined to be compatible with $\hat{T}^\nu_\mu = -S_{\mu\sigma} S^{\nu\sigma}$. This map is $O(1,3)$-equivariant

$$\mathbb{T}(lSl^T) = l\mathbb{T}(S)l^T, \quad \forall l \in O(1,3)$$

We replace ϕ by Φ, and $\mathbb{V}$ by $\mathbb{V}_4$ given as follows:

$$\Phi : Z(\mathbb{R}) \ni (X, \mathbf{v}, -\rho) \mapsto (XJ\mathbf{v}, -\rho) \in M(4,1,\mathbb{R})_{\mathrm{rk}=1} \times \mathbb{R}^\times,$$
$$\mathbb{V}_4 : \mathbb{R}^4 \times \mathbb{R}^\times \ni (\mathbf{w}, -\rho) \mapsto \rho \mathbf{w}\mathbf{w}^T \in \mathrm{Sym}(4,\mathbb{R})_{\mathrm{rk}\le 1}.$$

Theorem 2. $(\mathbb{T} \circ \mathbb{S})(X) = (\mathbb{V}_4 \circ \Phi)(X, \mathbf{v}, -\rho)$ *for all* $(X, \mathbf{v}, -\rho) \in Z(\mathbb{R})$.

Proof. $(\mathbb{T} \circ \mathbb{S})(X) = (XJX^T)g(XJX^T)^T = XJ\mathbb{G}(X)J^TX^T = -XJ\rho\mathbf{v}\mathbf{v}^TJ^TX^T$
$= \mathbb{V}_4((XJ\mathbf{v}, -\rho)) = (\mathbb{V}_4 \circ \Phi)((X, \mathbf{v}, -\rho))$. □

This theorem is illustrated as

$$\begin{array}{ccccc}
Z(\mathbb{R}) & \longrightarrow & M(4,2,\mathbb{R})_{\mathrm{rk}=2} \times S^1 \times \mathbb{R}^\times & \xrightarrow{\Phi} & M(4,1,\mathbb{R})_{\mathrm{rk}=1} \times \mathbb{R}^\times \\
\downarrow{\scriptstyle \tilde{\mathbb{V}}_2} & & \downarrow & & \downarrow{\scriptstyle \mathbb{V}_4} \\
Y(\mathbb{R}) & \longrightarrow & M(4,2,\mathbb{R})_{\mathrm{rk}=2} \xrightarrow{\mathbb{S}} \mathrm{Alt}(4,\mathbb{R})_{\mathrm{rk}=2} & \xrightarrow{\mathbb{T}} & \mathrm{Sym}(4,\mathbb{R})
\end{array}$$

3.4. Grassmann and Flag Manifold

$$\begin{array}{ccccc}
Y(\mathbb{R}) & \xleftarrow{\tilde{\mathbb{V}}_2} & Z(\mathbb{R}) & \xrightarrow{\Phi} & M(4,1,\mathbb{R})_{\mathrm{rk}=1} \times \mathbb{R}^\times \\
\downarrow & & \downarrow & & \downarrow \\
M(4,2,\mathbb{R})_{\mathrm{rk}=2} & \longleftarrow & M(4,2,\mathbb{R})_{\mathrm{rk}=2} \times S^1 \times \mathbb{R}^\times & \xrightarrow{\Phi} & M(4,1,\mathbb{R})_{\mathrm{rk}=1} \times \mathbb{R}^\times \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Grass}(4,2,\mathbb{R}) & \longleftarrow & \mathrm{Flag}(4;1,2,\mathbb{R}) & \longrightarrow & \mathrm{Grass}(4,1,\mathbb{R})
\end{array}$$

The flag manifold is realized as an incidence variety of the product of two Grassmann manifold:

$$\mathrm{Flag}(4;1,2,\mathbb{R}) = \{(V_1, V_2) \mid \dim V_1 = 1, \dim V_2 = 2, V_1 \subset V_2 \subset \mathbb{R}^4\}$$
$$\mathrm{Flag}(4;1,2,\mathbb{R}) = \{(V_1, V_2) \in \mathrm{Grass}(4,1,\mathbb{R}) \times \mathrm{Grass}(4,2,\mathbb{R}) \mid V_1 \subset V_2\}.$$

For $(X, \mathbf{v}) \in M(4,2,\mathbb{R})_{\mathrm{rk}=2} \times S^1$, two column vectors of X spans a two-dimensional subspace V_2, and a column vector $XJ\mathbf{v}$ generate a one-dimensional subspace V_1 in V_2. The map

$$\begin{array}{ccccc}
\mathrm{Grass}(4,2,\mathbb{R}) & \longleftarrow & \mathrm{Flag}(4;1,1,2,\mathbb{R}) & \longrightarrow & \mathrm{Grass}(4,1,\mathbb{R}) \\
V_2 & \leftarrow & (V_1, V_2) & \mapsto & V_1 \\
X & \leftarrow & (X, \mathbf{v}) & \mapsto & XJ\mathbf{v}
\end{array}$$

is the double fibration.

3.5. The Interpretation of the Off-Shell Condition

The vectors C and L in Clebsch parametrization should satisfy the following off-shell conditions [2]:

$$C_\nu C^\nu = 0, \quad L_\nu C^\mu = 0, \quad L_\nu L^\mu = -\rho. \tag{8}$$

We put $\bar{R} := \begin{pmatrix} 0 & 0 \\ 0 & -\rho \end{pmatrix}$. Then, the condition (8) is written as

$$\mathbb{G}(X) = \bar{R}.$$

In particular, in the case $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S^1$, we compute the maps $\mathbb{V}_2, \Phi$ and $\mathbb{V}_4$:

- $\mathbb{V}_2((\mathbf{v}, -\rho)) = -\rho\mathbf{v}\mathbf{v}^T = \bar{R}$,

- $(\mathbf{w}, -\rho) = \Phi((X, \mathbf{v}, -\rho)) = (XJ\mathbf{v}, -\rho) = (C, -\rho)$, this implies $\mathbf{w} = C$,
- $\mathbb{V}_4((\mathbf{w}, -\rho)) = \rho \mathbf{w}\mathbf{w}^T = \rho CC^T = \hat{T}$.

This coincides with the result in [2]. An unnatural J in the definition of Φ is for the sake of compatibility with the existing formula.

We now remark $R \in \mathrm{Sym}(2, \mathbb{R})_{\mathrm{rk}\leq 1}$.

$$\begin{array}{ccc} & & Z(\mathbb{R}) \\ & & \downarrow \\ \mathbb{G}^{-1}(\bar{R}) & \longrightarrow & Y(\mathbb{R}) \\ \downarrow & & \downarrow \mathbb{G} \\ \bar{R} & \in & \mathrm{Sym}(2, \mathbb{R})_{\mathrm{rk}=1} \end{array}$$

The group $GL(2, \mathbb{R})$ acts on $\mathrm{Sym}(2, \mathbb{R})_{\mathrm{rk}=1}$ and the stabilizer at $\bar{R}$ is a Borel subgroup

$$B = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \right\} \subset GL(2, \mathbb{R}).$$

Then, $\mathbb{G} : Y(\mathbb{R}) \longrightarrow \mathrm{Sym}(2, \mathbb{R})_{\mathrm{rk}=1}$ is a $GL(2, \mathbb{R})$-equivariant bundle. We regard the off-shell condition specifies a fiber of this bundle. A symmetry is hidden in the horizontal direction of this bundle, the group action of $GL(2, \mathbb{R})$. Of course, form the Clebsch parametrization point of view, the role of C and L is not the same; the off-shell condition specifies the special isotropic direction for C: the choice of this direction is controlled by the homogeneous space $GL(2, \mathbb{R})/B$.

4. Discussion

We describe the symmetry of equations of dressed photon in a general manner. The tensor S is understood as an affine version of Plücker coordinates of Grassmann manifold $\mathrm{Grass}(4, 2, \mathbb{R})$. The splitting expression of the tensor $\hat{T}$ is related with an affine version of flag manifold $\mathrm{Flag}(4; 1, 2, \mathbb{R})$. We find the off-shell condition (8) chooses the special fiber of the homogeneous bundle. This mathematical interpretation of the choice may have a physical interpretation, especially in the context of Clebsch variables, however, which must be a future work. We also remark that the existence of the symmetry in arbitrary dimension suggests a feedback from the theory of dressed photon to the theory of reductive dual pairs on the pullback of nilpotent orbits [6], which is also a topic of future study.

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