

Conserved relativistic Ertel's current generating the vortical and thermodynamic aspects of space-time

Hirofumi Sakuma,^{*,§} Izumi Ojima,^{*,¶} Hayato Saigo^{†,||} and Kazuya Okamura^{*,‡,**}

^{*}Research Origin for Dressed Photon
3-13-19 Moriya-cho, Kanagawa-ku
Yokohama, Kanagawa 221-0022, Japan

[†]Nagahama Institute of Bio-Science and Technology
1266 Tamura, Nagahama, Shiga 526-0829, Japan

[‡]Graduate School of Informatics, Nagoya University
Chikusa-ku, Nagoya 464-8601, Japan

[§]sakuma@rodrep.or.jp

[¶]ojima@gaia.eonet.ne.jp

^{||}h_saigo@nagahama-i-bio.ac.jp

^{**}k.okamura.renormalizable@gmail.com

Received 17 March 2022

Revised 29 July 2022

Accepted 23 August 2022

Published 28 September 2022

Motivated by Aoki *et al.*'s recent research on conserved charges and entropy current, we reinvestigated the conservation of relativistic Ertel's current, which has received little attention outside the field of geophysical fluid dynamics. Ertel's charge is an important indicator of the correlation between vortex vectors and entropy gradient fields in Earth's meridional heat transport. We first show that in the generalized Hamiltonian structure of baroclinic fluids, the duality between the total energy and the Casimir as a function of Ertel's charge plays an important role in the nonrelativistic case. Then, by extending the result to relativistic cases, we show that this finding has far-reaching implications not only for space-time issues in cosmology but also for the foundation of quantum field theory. An especially important finding is that, as an unreported dual form of the Einstein field equation, we identify a special equation satisfied not only by the vortex tensor field generated by the conserved charge but also by the Weyl tensor in interpreting the physical nature of the metric tensor $g^{\mu\nu}$, which appears in the cosmological term $\Lambda g^{\mu\nu}$.

Keywords: Ertel's charge; entropy; generalized Hamiltonian; cosmological term; Weyl tensor.

PACS numbers: 04.20.Cv, 04.40.Nr, 05.90.+m, 47.10.ab

[§]Corresponding author.

This is an open access article published by World Scientific Publishing Company. It is distributed under the terms of the [Creative Commons Attribution 4.0 \(CC BY\) License](https://creativecommons.org/licenses/by/4.0/), which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Two key issues are addressed in this paper: first, we shed new light on the peculiarity of the gravitational field, which has, in addition to its well-documented geometrical nature, an unexpected thermodynamic nature;^{1–4} second, we clarify the main cause of the inherent defect in the present form of quantum field theory (QFT), exemplified by Haag’s no-go theorem⁵ in axiomatic QFT. As we show below, these two seemingly disparate concepts are inextricably linked by the important dynamical role of the spinor field (or the vortical field in the case of classical mechanics), which connects a given physical system to the surrounding space–time as its dynamical environment.

Many issues in nonrelativistic physics require the use of an abstract space to characterize the time-dependent dynamic behaviors of a given physical system, such as the Hilbert space \mathfrak{H} in quantum mechanics and phase space in classical Hamiltonian (H) systems. In general, these spaces are not related to the physics of a given system. The fact that this condition changes drastically in relativistic field theory is well-known and has become common knowledge in physics, while the well-known *wave–particle duality* in quantum mechanics remains a mystery. However, once it is accepted that space–time is a physical entity in which “spacelike” corresponds to wavelike entities, such as the spacelike momentum field required for quantum field interactions (Greenberg–Robinson (GR) theorem^{6,7}), while “timelike” corresponds to localized particle-like entities, then the wave–particle duality of quantum entities appears to be a natural consequence of the fact that quantum entities always coexist with embedded space–time; in other words, quantum entities are dynamically integrated with space–time as their environment.

Dirac’s finding has played an important role in considering the physicality of space–time, revealing that quantum spin is a feature of relativistic space–time. Since then, the idea that space–time is composed of specific types of spin networks has been investigated, most notably by Penrose⁸ and the many researchers that developed loop quantum gravity (LQG) theory. Therefore, an essential question associated with the space–time conundrum is how the thermodynamic properties of space–time fit with this spin network concept. The peculiarity of the space–time problem in relativistic scenarios lies in the fact that space–time not only has a mathematical meaning, which can be used to represent a given physical system, but is also a physical field that must be represented by itself. To explain this reciprocal nature which can be regarded as the interdependent duality (ID) existing between a given physical system and its associated space–time, we must identify the emergent processes of this “physical” space–time.

As an informative example of the ID mentioned above, we briefly refer to micro–macro duality (MMD) theory,⁹ which, as briefly explained in a recent review paper¹⁰ on nanophotonics, *can be regarded as a modern version of “quantum–classical correspondence”*; *this theory was rigorously derived based on the pertinent generalization of the superselection rule, with the sectors originally formulated by Doplicher–Haag–Roberts.*^{11,12} Any efforts to explore unseen microscopic quantum

worlds require experiments that investigate the dynamical interactions between micro-quantum and macro-classical worlds. This simple fact clearly shows that our descriptions of unseen microscopic worlds depend heavily on the “vocabulary” we use in classical physics. Therefore, in the metaphorical sense, classical physics operates not as a static “space” onto which the microscopic world is projected, but rather as an interacting dynamic “space–time” which we use to describe the targeted microscopic world.

This viewpoint is supported as a key element of MMD theory. Among the many noteworthy accomplishments of MMD theory, the following two theoretical results are particularly relevant to the discussion in this paper. First, due to its infinite degrees of freedom, QFT inevitably involves a combination of quantum and classical fields, with the latter emerging due to the existence of disjoint (refined notion of unitary nonequivalent) generalized sectors with nontrivial factor representations acting as order parameters of the emerging classical field. The irrelevance of Schrödinger’s cat thought experiment can be easily clarified¹⁰ by the basic results of MMD theory. Second, with the exception of the different algebraic structures of certain physical quantities in their respective fields, i.e. anti-commutativity versus commutativity, we can consider both fields in a unified framework.

The main discussions begin in Sec. 2 with classical physics; however, we show that even in classical fluid dynamics systems, there exists an intriguing ID structure that eventually clarifies the discussed space–time peculiarity. In Secs. 2 and 3, by reviewing the basic structures of barotropic and baroclinic fluids, as well as their generalized Hamiltonian (H) structures, we identify a key conserved dynamical quantity that generates the above-mentioned ID structure. The reason we focus on this conserved quantity is because we found that recent works by Aoki *et al.*,^{13,14} particularly on the conservation of entropy in general relativity, are quite informative in the present research. Then, in Sec. 4, we show how the main aims of our paper, as stated in the first paragraph of this introductory section, are achieved based on the results of the previous section. In the final section, we present our conclusions and some novel perspectives on QFT and cosmology.

2. Dynamics of Barotropic and Baroclinic Fluids and H Structures

We begin with the well-known equation of motion of a perfect fluid in nonrelativistic fluid mechanics:

$$D_t v_\mu := \partial_t v_\mu + v^\nu \partial_\nu v_\mu = -\frac{1}{\rho} \partial_\mu p, \quad (1)$$

where the notations are conventional. A fluid is classified as barotropic or baroclinic based on its form of $\partial_\mu p/\rho$; that is, if $p = p(\rho)$ or $\rho = \rho_0 = \text{const}$, the fluid is barotropic; otherwise, the fluid is baroclinic. In other words, the barotropic or baroclinic nature of the fluid is characterized by whether $\partial_\mu p/\rho$ is a conservative force field, which has a decisive influence on whether the associated vorticity field is conservative. Thus, we refer to the *Lagrange’s vortex theorem for barotropic flows*, which shows that vortices are free of generation and extinction.

Since we are primarily interested in vorticity and entropy fields, it is useful to rewrite the baroclinic form of Eq. (1) in terms of the vorticity $\zeta_{\mu\nu}$ and specific (i.e. per unit mass) entropy s fields by using the first law of thermodynamics (Eq. (2)) and the vector identity given in Eq. (3):

$$-dp/\rho = Tds - dw, \quad \text{with } \partial_t s + v^\nu \partial_\nu s = 0, \quad (2)$$

$$v^\nu \partial_\nu v_\mu = v^\nu (\partial_\nu v_\mu - \partial_\mu v_\nu) + \partial_\mu (v^\nu v_\nu/2), \quad (3)$$

where T and w are the absolute temperature and specific enthalpy, respectively. From Eqs. (2) and (3), Eq. (1) becomes

$$\partial_t v_\mu + \partial_\mu (w + v^\nu v_\nu/2) - \zeta_{\mu\nu} v^\nu = T \partial_\mu s. \quad (4)$$

The most well-known example of a baroclinic fluid is the atmosphere, for which the ideal gas law can be applied with a high degree of accuracy. The atmosphere is particularly important in our discussion because it provides a useful fluid dynamic system with a nonuniform entropy distribution in both the vertical and meridional directions; furthermore, energetic vortical fields known as baroclinic eddies play important dynamical roles in heat transport along the meridional direction. In the dynamics of this heat transport, there is a strong correlation between the vorticity ζ and entropy gradient ∇s fields, which can be described by Ertel's potential vorticity Q ,¹⁵ which is defined as

$$Q := \frac{1}{\rho} (\zeta \cdot \nabla s), \quad \partial_t Q + v^\nu \partial_\nu Q = 0. \quad (5)$$

The above is the most important conserved quantity in the field of geophysical fluid dynamics.

It should be noted that the importance of baroclinicity in the atmosphere varies with scale. In general, in typical laboratory experiments using air, such as wind tunnel studies, air flows behave as barotropic flows since the entropy gradient is negligible; however, for air flows with horizontal scales greater than several 100 or a few 1000 km, baroclinicity becomes a nonnegligible dynamical factor. In Sec. 1, we discussed MMD theory, which indicates that the actual world in which we live consists of an ID structure that bridges quantum and classical physics, rather than *the prevailing view that the laws of classical physics are not fundamental but are approximated from "genuine fundamental" quantum mechanics*. We believe that the scale dependency of baroclinicity, represented by Q in the comparison of barotropic and baroclinic flows, is analogous to that of the Planck constant h in the comparison of quantum and classical physics in MMD theory. To show that this resemblance is not superficial but rather has essential implications for the main issue in this paper, we investigated the generalized H structure of baroclinic flows.

The generalized H structure was derived from the so-called noncanonical form of the H formulation, with the seminal work initiated by Arnol'd¹⁶ and further developed by a group of applied mathematicians and physicists.¹⁷⁻¹⁹ For a baroclinic perfect fluid dynamics system with Eulerian representation, the generalized H ,

denoted by H_G , has the form²⁰

$$H_G := E + C_F, \quad E := \int \langle \rho v^\nu v_\nu / 2 + \rho e(\rho, s) \rangle dV, \quad C_F := \int \langle \rho F(s, Q) \rangle dV, \quad (6)$$

where E and C_F are the total energy with $e(\rho, s)$ being the internal energy density, a Casimir constructed by an arbitrary function F of s and Q . Since the given fluid dynamical system can be described by five independent variables, namely, v^μ ($1 \leq \mu \leq 3$) and two thermodynamical variables, we choose ρ and s as the thermodynamic variables because Q is expressed in terms of these two variables. First, when we compare the conservative quantity C_F with the total energy E , we can observe that it is not merely an additional constant of motion. This occurs because both E and C_F are “complete” in the sense that they include all five variables. In other words, they are equal pairs of “complete” constants of motion.

The significant advantage of H_G over E becomes clear when we consider the stability of a given steady state of the fluid because any given steady state of the baroclinic flow can be represented by the condition that the first variation in H_G vanishes; that is,

$$\delta H_G = \delta E + \delta C_F = 0, \quad (7)$$

which can be rewritten as the combination of the steady-state version of Eq. (4) and

$$F - Q \frac{\partial F}{\partial Q} + B(s, Q) = 0. \quad (8)$$

Equation (8) shows that an arbitrary function F can be determined by Bernoulli's function $B(s, Q)$, which characterizes the given steady state. Then, we can demonstrate the formal stability of the state²¹ (the stability of a given steady state for infinitesimally small amplitude perturbations with arbitrary forms) if the second variation in H_G is sign definite. In our discussion of the ID structure, a particularly important aspect of Eq. (7) is that the balance between δE and δC_F can be regarded as a unique “interaction” between two dynamics with and without explicit forms of Q . It is clear that the phase-space trajectory of a linearly unstable mode, such as those represented by one of the separatrices of H_G , i.e. $\delta^2 H_G = 0$, cannot be described without C_F . Thus, we believe that perturbation methods with only the Hamiltonian as the total energy are not generally adaptable since, as our present discussion shows, these methods cannot cover the dynamic behaviors of certain unstable modes. We also believe that the root cause of Haag's no-go theorem in axiomatic QFT, as discussed in Sec. 1, can be attributed to such an inability inherent in the conventional Hamiltonian approaches. To further show the decisive role of Q in ID structures, we next examine the relativistic expression of Q .

3. Converted Form of the Relativistic Equation of Motion

First, we fix the sign convention as $(+, -, -, -)$ and introduce a nondimensional four-velocity vector u^μ that satisfies the normalization condition $u^\nu u_\nu = 1$. In Secs. 3

and 4, for simplicity, unless otherwise stated, we develop our arguments within the framework of special relativity. The following basic arguments on the relativistic equations from Eqs. (9)–(15) are given by Landau and Lifshitz²² (pp. 506–508). Regarding the first law of thermodynamics, we have

$$-\frac{dp}{n} = Td\left(\frac{\sigma}{n}\right) - d\left(\frac{w}{n}\right), \tag{9}$$

where n denotes the “particle number” corresponding to the density ρ in the non-relativistic case, and σ/n and w/n are the specific entropy and enthalpy, respectively, as in Eq. (2). The relativistic equation of continuity is given by

$$\partial_\nu(nu^\nu) = 0, \tag{10}$$

and the energy–momentum tensor for a perfect fluid has the following form:

$$T^{\mu\nu} = wu^\mu u^\nu - pg^{\mu\nu}. \tag{11}$$

The tensor divergence of Eq. (11) gives the following equations of motions:

$$\partial_\nu T_\mu^\nu = u_\mu \partial_\nu(wu^\nu) + wu^\nu \partial_\nu u_\mu - \partial_\mu p = 0. \tag{12}$$

The projection of Eq. (12) in the direction u^μ can be calculated by $u^\mu \partial_\nu T_\mu^\nu = 0$; with Eq. (9), this becomes

$$u^\nu \partial_\nu(\sigma/n) = 0, \tag{13}$$

which corresponds to the second equation in Eq. (2). Next, we calculate the component of $\partial_\nu T_\mu^\nu = 0$ perpendicular to u^μ as

$$\partial_\nu T_\mu^\nu - u_\mu u^\nu \partial_\sigma T_\nu^\sigma = 0, \tag{14}$$

which yields

$$wu^\nu \partial_\nu u_\mu - \partial_\mu p + u_\mu u^\nu \partial_\nu p = 0. \tag{15}$$

This equation corresponds to the nonrelativistic form of $\rho D_t v_\mu = 0$ in Eq. (1).

As we have noted with Eq. (4) in Sec. 2, in terms of the vorticity and the entropy field, the form of Eq. (4) is preferable to the form of Eq. (1). The derivation of the relativistic form of Eq. (4) was given by Lichnerowicz²³ (p. 30) and it has the following form:

$$\omega_{\mu\nu} u^\nu = T \partial_\mu(\sigma/n), \quad \omega_{\mu\nu} := \partial_\mu[(w/n)u_\nu] - \partial_\nu[(w/n)u_\mu]. \tag{16}$$

For the sake of readers’ convenience, in Appendix A, we give the details of this derivation. Note that Eq. (13) is included in Eq. (16), as shown by $0 = u^\mu \omega_{\mu\nu} u^\nu = T u^\mu \partial_\mu(\sigma/n)$. We can also observe that Eq. (16) remains valid for curved space–time if we replace ∂_μ with a covariant derivative, denoted as ∇_μ .

4. Ertel's Charge as the Source of an Entropic Vortex Field

To appreciate the importance of Eq. (16), we use an explicit (writing down all the elements) matrix representation of Eq. (17) below to derive the conserved Ertel's current.

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & -\omega_{31} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & \omega_{31} & -\omega_{23} & 0 \end{pmatrix}. \quad (17)$$

First, after defining the pseudoscalar Ω in Eq. (18), we introduce ${}^*\omega^{\mu\nu}$, which is the Hodge dual of $\omega_{\mu\nu}$, i.e.

$$\Omega := \omega_{01}\omega_{23} + \omega_{02}\omega_{31} + \omega_{03}\omega_{12}, \quad (18)$$

$${}^*\omega^{\mu\nu} = \begin{pmatrix} 0 & -\omega_{23} & -\omega_{31} & -\omega_{12} \\ \omega_{23} & 0 & -\omega_{03} & \omega_{02} \\ \omega_{31} & \omega_{03} & 0 & -\omega_{01} \\ \omega_{12} & -\omega_{02} & \omega_{01} & 0 \end{pmatrix}. \quad (19)$$

From Eqs. (17) and (19), we obtain

$${}^*\omega^{\mu\kappa}\omega_{\kappa\nu} = \Omega g^\mu_\nu, \quad {}^*\omega^{\mu\nu}\omega_{\nu\mu} = 4\Omega. \quad (20)$$

According to Eq. (16), we have

$$({}^*\omega^{\mu\kappa}\omega_{\kappa\nu})u^\nu = \Omega g^\mu_\nu u^\nu = T({}^*\omega^{\mu\kappa})\partial_\kappa(\sigma/n); \quad (21)$$

thus, we can obtain

$$\Omega_T u^\mu = {}^*\omega^{\mu\kappa}\partial_\kappa(\sigma/n), \quad \text{where } \Omega_T := \Omega/T. \quad (22)$$

By substituting Eq. (19) into Eq. (22) and with a series of manipulations based on the skew symmetry of $\omega_{\mu\nu}$, we can finally derive that

$$\Omega_T \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} -\partial_1[\omega_{23}(\sigma/n)] - \partial_2[\omega_{31}(\sigma/n)] - \partial_3[\omega_{12}(\sigma/n)] \\ \partial_0[\omega_{23}(\sigma/n)] - \partial_2[\omega_{03}(\sigma/n)] + \partial_3[\omega_{02}(\sigma/n)] \\ \partial_0[\omega_{31}(\sigma/n)] + \partial_1[\omega_{03}(\sigma/n)] - \partial_3[\omega_{01}(\sigma/n)] \\ \partial_0[\omega_{12}(\sigma/n)] - \partial_1[\omega_{02}(\sigma/n)] - \partial_2[\omega_{01}(\sigma/n)] \end{pmatrix}. \quad (23)$$

Based on this expression, we can observe that

$$\partial_\nu(\Omega_T u^\nu) = 0. \quad (24)$$

In addition, using Eq. (16) again, we have

$$-\Omega u^0 = T[\omega_{23}\partial_1(\sigma/n) + \omega_{31}\partial_2(\sigma/n) + \omega_{12}\partial_3(\sigma/n)]; \quad (25)$$

thus, we obtain

$$\Omega_T = \Omega_Q := -[\omega_{23}\partial_1(\sigma/n) + \omega_{31}\partial_2(\sigma/n) + \omega_{12}\partial_3(\sigma/n)]/u^0, \quad (26)$$

which is the relativistic expression of Ertel’s potential vorticity Q given in Eq. (5). Following the convention of theoretical physics, we refer to Ω_T as Ertel’s charge. The conservation property of Eq. (24) can be extended to curved space–time by substituting ∂_μ in Eq. (23) with the covariant derivative ∇_μ and using the tensor identity

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\omega_{\kappa\lambda} = -R^\sigma_{\kappa\mu\nu}\omega_{\sigma\lambda} - R^\sigma_{\lambda\mu\nu}\omega_{\kappa\sigma} \tag{27}$$

to calculate the vector divergence on the right-hand side of Eq. (23), where $R^\alpha_{\beta\gamma\delta}$ denotes the Riemann curvature tensor.

Remark 1 A relativistic extension of Ertel’s potential vorticity was first done by Katz²⁴ based on the converted form of a relativistic equation of motion derived by Lichnerowicz referred to at the end of Sec. 3. In his derivation, as in the case of ours, Katz first introduces ${}^*\omega^{\mu\nu}$ (Hodge dual of $\omega_{\mu\nu}$ defined in Eq. (19)). Then, using it, he defines a “vorticity four-vector” $\hat{\omega}^\mu$ of the form

$$\hat{\omega}^\mu = {}^*\omega^{\mu\nu}u_\nu. \tag{28}$$

The conservation law he shows turns out to be

$$\partial_\nu(\Omega_K u^\nu) = 0, \quad \Omega_K := \hat{\omega}^\nu \partial_\nu(\sigma/n), \tag{29}$$

where we have simplified his derivation by confining ourself solely to the case of special relativity. Using Eq. (10), the above equation can be rewritten as

$$u^\nu \partial_\nu[\Omega_K/n] = 0. \tag{30}$$

Based on this advective expression, Katz simply points out that the nonrelativistic limit of Ω_K/n is Q given in Eq. (5). Note, however, that Ω_K in Eq. (29) is defined as the inner-product of four vectors $\hat{\omega}^\nu$ and $\partial_\nu(\sigma/n)$, while the numerator of Q in Eq. (5) is expressed as the inner-product of nonrelativistic three vectors of ζ and ∇s . In this respect, we do not think that Katz’s statement on the nonrelativistic limit is trivial and therefore needs further explanation. In order to prove his statement as well as the equivalence of Katz’s and our derivations, we first show that, by direct calculations using Eqs. (16), (19), (26) and (28), we have

$$\Omega_K = \Omega_T (= \Omega_Q). \tag{31}$$

Thus, by comparing Ω_Q/n in the advection equation $u^\nu \partial_\nu[\Omega_Q/n] = 0$ and Q in Eq. (5), we readily see that Q gives the nonrelativistic limit of $\Omega_K/n = \Omega_T/n$.

The advantage of the newly derived expression Ω_T over Ω_K is that the conserved current $\Omega_T u^\mu$ can be shown to be an entropy current tied closely to “space–time dynamics” now we are considering in this paper. According to the second equation in Eq. (16), the physical dimension of Ω in Eq. (18), denoted by $\dim[\Omega]$, becomes $\dim[\Omega] = l^{-2} \dim[(w/n)^2]$, where l denotes the length scale. Since n and w are the particle number and the energy per unit volume, respectively, if we use a natural unit system, then we have $\dim[n] = l^{-3}$ and $\dim[w^2] = l^{-8}$. Thus, it turns out that

$\dim[\Omega] = l^{-1}/l^3$, indicating that $\dim[\Omega] = \dim[T^{\mu\nu}]$ in Eq. (11) and hence, from Eq. (9), the physical dimension of Ω_T is the entropy per unit volume.

A particularly intriguing property of $\omega_{\mu\nu}$ is that “Dirac’s γ matrix” $\hat{\gamma}^{\mu\nu}$ can be constructed from Eq. (20). In fact, if we define $\hat{\gamma}_\nu^\mu$ such that $\hat{\gamma}_\nu^\mu := (*\omega^{\mu\sigma})\omega_{\sigma\nu}$, then, according to Eq. (20), we get $\hat{\gamma}_\nu^\mu = \Omega g_\nu^\mu$. By raising the suffix ν , we have that $\hat{\gamma}^{\mu\nu} = \Omega g^{\mu\nu}$; thus, we find that

$$\frac{1}{\Omega} (\hat{\gamma}^{\mu\nu} + \hat{\gamma}^{\nu\mu}) = 2g^{\mu\nu}, \tag{32}$$

which is the well-known anti-commutation relation. To further examine the implications of Eq. (32), we investigated the relation between $g^{\mu\nu}$ and $*\omega^{\mu\sigma}\omega_\sigma^\nu/\Omega$. According to Eq. (20), $g^{\mu\nu}$ can be rewritten as follows:

$$g^{\mu\nu} = \frac{* \omega^{\mu\sigma} \omega_\sigma^\nu}{\Omega} = \frac{* \omega^{\mu\sigma} \omega_\sigma^\nu (* \omega^{\kappa\lambda} \omega_{\lambda\kappa})}{\Omega (* \omega^{\kappa\lambda} \omega_{\lambda\kappa})} = \frac{* \omega^{\mu\sigma} \omega_\sigma^\nu (* \omega^{\kappa\lambda} \omega_{\lambda\kappa})}{(* \omega^{\kappa\lambda} \omega_{\lambda\kappa})^2 / 4}. \tag{33}$$

Recall that, in general, $g^{\mu\nu}$ is not a physical quality but rather a purely mathematical quantity. However, there exists an exceptional case in which $g^{\mu\nu}$ becomes physical, as shown by Eq. (34), which was derived by lengthy straightforward calculations^{25,26} on the Weyl conformal tensor $W_{\alpha\beta\gamma\delta}$.

$$W^{\mu\alpha\beta\gamma} W_{\alpha\beta\gamma}^\nu - \frac{1}{4} W^2 g^{\mu\nu} = 0, \quad W^2 := W^{\alpha\beta\gamma\delta} W_{\alpha\beta\gamma\delta}. \tag{34}$$

Equation (34) shows that, for nonvanishing W^2 , the cosmological term $\Lambda g^{\mu\nu}$ can be interpreted not as vacuum but as (conformal) gravitational energy–momentum tensor. By comparing Eq. (33) with Eq. (34), we find that

$$g^{\mu\nu} = \frac{W^{\mu\alpha\beta\gamma} W_{\alpha\beta\gamma}^\nu}{W^2 / 4} = \frac{* \omega^{\mu\sigma} (* \omega^{\kappa\lambda}) \omega_\sigma^\nu \omega_{\kappa\lambda}}{(* \omega^{\kappa\lambda} \omega_{\lambda\kappa})^2 / 4} = \frac{(* \omega^{\mu\sigma}) * \omega^{\kappa\lambda} \omega_\sigma^\nu \omega_{\kappa\lambda}}{(4\Omega)^2 / 4}, \tag{35}$$

which clearly shows that $(4\Omega)^2$ correlates directly with W^2 .

As we have referred to at the end of Sec. 1, quite recently, Aoki *et al.* proposed novel mathematical schemes defining precisely a conserved quantity in a curved space–time, that can be applied not only to the energy and momentum for matter but also to the entropy of a given system. And in order to justify the new schemes, they have successfully conducted various verifications for different gravitational systems. We think that one of their new schemes on the entropy conservation provides tantalizing new information revealing the important role played by Weyl tensor in gravitational “entropy dynamics”.

A possible connection between gravitational entropy and Weyl tensor was suggested by Penrose in terms of *Weyl curvature hypothesis*²⁷ which may explain the observed extremely isotropic space–time structure of our early universe. Through their series of studies, of which brief overview is given in the subsequent section, Sakuma *et al.*^{25,28} are now pursuing a new study on developing an *extended dark matter* (EDM) model having a form of cosmological term: $\lambda_{\text{dm}} g_{\mu\nu}$ where the metric tensor $g_{\mu\nu}$ is related to Weyl tensor field through Eq. (35). The main reason why we

consider an EDM is because we conjecture that, due to the ID structure arising from material fields and their environmental space–time, dark matter phenomena would not be figured out solely in terms of timelike particle dynamics, though *the timelike entity associated with EDM* is our primary concern. We think that the aforementioned studies by Aoki *et al.*¹⁴ are helpful in identifying this timelike entity, because they have proposed a general form of the definition of a timelike entropy current. The key concept in their definition is, what they call, *timelike intrinsic vector* $\hat{\zeta}^\mu$ satisfying

$$T_\nu^\mu \nabla_\mu \hat{\zeta}^\nu = 0 \Rightarrow (\nabla_\mu \hat{\zeta}^\mu = 0, \text{ for } T_\nu^\mu = \lambda_{\text{dm}} g_\nu^\mu), \quad (36)$$

where ∇_μ and T_ν^μ , respectively, denote covariant derivative and the energy–momentum tensor under consideration, the latter of which becomes $\lambda_{\text{dm}} g_\nu^\mu$ in our present case.

From Eqs. (10) and (13), we have the well-known equation: $\nabla_\nu(\sigma u^\nu) = 0$. Interestingly enough, we have already shown that Ω_Q in Eq. (26) whose physical dimension is exactly the same as that of σ also satisfies the same equation of $\nabla_\nu(\Omega_Q u^\nu) = 0$, though Ω_Q is composed of a certain vortical field while σ is not such a kind of quantity. Since the physical meaning of $\nabla_\nu(\Omega_Q u^\nu) = 0$ must be understood in terms of Eq. (36), we can say that, thanks to it, $\Omega_Q u^\nu$ is a timelike gravitational entropy current associated with Weyl curvature field. A novel proposal of dark matter model based on this timelike gravitational entropy current will be given in Subsec. 5.2.1.

From the definition of Ω_Q in Eq. (26), we see that it is a classical spin field generated by the two-dimensional vortical motion of a fluid particle (or element) confined on a comoving isentropic surface. The equality of the following two expressions $u^\nu \partial_\nu(\sigma/n) = u^\nu \partial_\nu(\Omega_Q/n) = 0$ derived from Eqs. (13) and (30) holds if there exists such a function f as to satisfy $\Omega_Q/n = f(\sigma/n)$. Actually, it is known that this equality holds for a wider class of relation between Ω_Q/n and σ/n called *the fluid element relabeling symmetry* (FERS)²⁹ leading to the invariance of Casimir functionals referred to in Sec. 2. Thus, as we have stated at the beginning of the abstract of this paper, our reinvestigation on relativistic extension of Ertel’s potential vorticity having the property of FERS was set out with the motivation of aiming at a meaningful connection between Aoki *et al.*’s¹⁴ novel achievement of Eq. (36) and our central result of Eq. (35).

5. Brief Conclusions and Novel Perspectives on QFT and Cosmology

5.1. Conclusions on Ω_T dynamics and its implication for QFT

By reinvestigating the relativistic form of Ertel’s charge Ω_T , of which importance has been largely ignored except in the field of geophysical fluid dynamics, we found that $\Omega_T u^\mu$ is a conserved “entropy current” in the sense of FERS mentioned above. The importance of this finding is that while the physical dimension of Ω_T is the entropy per unit volume, this quantity is not identical to the thermodynamic entropy density;

instead, it is related to both the vortical modes of a given energy–momentum field $T^{\mu\nu}$ in Eq. (11) and the associated space–time $g^{\mu\nu}$ (defined within the framework of conformal gravity (34)). As a result, Ertel's charge Ω_T plays an important role in the ID structure discussed in Sec. 1.

We think that the category-theoretic perspective plays a crucial role in exploring the nature of Ertel's charge in terms of QFT. In thermodynamics, the concept of entropy is understood through the order structure between thermodynamic states and the state transitions between them. Categories are a generalization of both the order and the group-theoretic structure, and they are helpful in capturing the essence of the spatio-temporal structure. In this respect, Saigo³⁰ proposed the idea of considering the category algebra, which is a noncommutative convolution algebra defined on a spatio-temporal category; furthermore, he considered the states as linear functions on it, as quantum fields and their states. We believe that by reformulating the discussion of Ertel's charge from this perspective, we can naturally “quantize” the contents of this paper. In the context of the quantized theory, the (possibly continuous) sector structure arises, which describes the macroscopic nature of the theory. As we have referred to in the brief explanation of MMD theory given in Sec. 1, so-called *order parameters* distinguish different sectors which have been shown to be treatable in recent quantum measurement theory.³¹ We believe that, as a future challenge, it is important to reexamine the role of Ertel's charge from the viewpoint of sector theory.

5.2. Novel perspective on cosmology

5.2.1. On a dark matter model

In Sec. 4, we showed that $\dim[\Omega] = \dim[T^{\mu\nu}]$ in Eq. (11), and we found that the nonzero value of Ω^2 corresponds to the nonzero value of W^2 , which suggests that the nonzero Ω is associated with a special energy field associated with nonzero W^2 . In general, in the Einstein field equation (37)

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu}, \quad (37)$$

energy–momentum fields are associated directly with Ricci curvature terms; thus, a peculiar energy field such as Ω would be related to the cosmological term $\Lambda g^{\mu\nu}$ we discussed with Eq. (34) to some (or a large) extent. As the first step toward understanding the physical meaning of a conserved “entropy” density $\Omega_T (= \Omega/T)$, we introduce a constant reference temperature T_R , the magnitude of which is immaterial at this point in our discussion but which will become important in our newly proposed hypothesis known as simultaneous conformal symmetry breaking (SCSB) of electromagnetic and gravitational fields. With T_R , we can introduce a nondimensional parameter such as the particle number $\tilde{n} := T_R/T$, which is inversely proportional to the temperature T . With \tilde{n} , we can rewrite Eq. (24) as

$$\nabla_\nu(\tilde{n}\Omega u^\nu) = 0. \quad (38)$$

Therefore, if we redefine $\tilde{\Omega}$ as $\tilde{\Omega} := \tilde{n}\Omega$ and introduce $\tilde{T}^{\mu\nu}$ as $\tilde{T}^{\mu\nu} := \tilde{\Omega}u^\mu u^\nu$, we can obtain

$$\nabla_\nu \tilde{T}^{\mu\nu} = 0, \tag{39}$$

since u_μ satisfies the geodesic condition $u^\nu \nabla_\nu u_\mu = 0$ at the galactic scale. Note that the nonzero $\tilde{\Omega}$ does not correspond to the nonzero Ricci scalar curvature R , but instead corresponds to the nonzero Weyl curvature W^2 ; thus, the nonzero current $\tilde{\Omega}u^\mu$ can exist even in “nearly vacuum” regions where $R^{\mu\nu} \approx 0$. Furthermore, since the magnitude of $\tilde{\Omega}$ is inversely proportional to T , $\tilde{\Omega}$ would become extremely small soon after the beginning phase of the big bang and then increases, which suggests that the current $\tilde{\Omega}u^\mu$ is in a region where $R^{\mu\nu} \approx 0$ is a promising candidate for the (cold) dark matter model.

5.2.2. *Brief review of previous cosmological studies by Sakuma et al.*

In their recent study on the off-shell properties of quantum fields motivated by enigmatic dressed photon research,¹⁰ Sakuma et al.^{25,28} shed new light on the long-forgotten GR theory (proved in axiomatic QFT) referred to in Sec. 1, which states that interactions among quantum fields must inevitably accompany spacelike momentum supports. Since these findings revealed a novel perspective on cosmology, especially for models of dark energy and dark matter, we will briefly review these findings before we discuss the above-mentioned dark matter model $\tilde{\Omega}u^\mu$ further. First, on the basis of the GR theorem, in previous research, the mathematical form of a spacelike electromagnetic field, which can be regarded as the extension of the charge-free Maxwell equation into spacelike momentum domains, was investigated. The main conclusions can be summarized as follows ((i)–(iv)):

- (i) The extended electromagnetic four-vector potential U_μ can be represented by the Clebsch parametrization (CP)³² with the parameters (λ, ϕ) ; the former satisfies the spacelike Klein–Gordon (KG) equation $\nabla_\sigma \nabla^\sigma \lambda - (\kappa_0)^2 \lambda = 0$, where κ_0 is the experimentally determined dressed photon constant, while the latter satisfies either the same KG equation or $\nabla_\sigma \nabla^\sigma \phi = 0$, depending on whether U_μ is spacelike or lightlike. The lightlike U_μ can be interpreted as a $U(1)$ gauge boson, while the spacelike U_μ provides the necessary spacelike momentum supports for field interactions. In fluid mechanics, CP is used for canonical H formulations of barotropic fluids. CP is suitable for extended free Maxwell fields because, in sharp contrast to Eq. (18), for the baroclinic case, the pseudoscalar $\Omega_{(ro)}$ defined by Eq. (40) always vanishes:

$$\Omega_{(ro)} := S_{01}S_{23} + S_{02}S_{31} + S_{03}S_{12} = 0, \tag{40}$$

where $S_{\mu\nu} := \nabla_\mu U_\nu - \nabla_\nu U_\mu$ denotes the field strength of the extended electromagnetic field. According to Eq. (40), as in the case of a free electromagnetic wave, the extended “electric” and “magnetic” fields are perpendicular to each other.

- (ii) The energy–momentum tensor $\hat{T}^{\mu\nu}$ for both cases can be written in a unified form as

$$\hat{T}^{\mu\nu} = \hat{S}_\sigma^{\mu\nu\sigma} - \frac{1}{2}\hat{S}_{\alpha\beta}^{\alpha\beta}g^{\mu\nu}, \quad \hat{S}_{\alpha\beta\gamma\delta} := S_{\alpha\beta}S_{\gamma\delta}. \quad (41)$$

Note that due to the skew-symmetric nature of $S_{\mu\nu}$, $\hat{S}_{\alpha\beta\gamma\delta}$ satisfies exactly the same properties as the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$; that is,

$$R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}, \quad R_{\alpha\beta\delta\gamma} = -R_{\alpha\beta\gamma\delta}, \quad R_{\gamma\delta\alpha\beta} = R_{\alpha\beta\gamma\delta}, \quad (42)$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0. \quad (43)$$

Equation (43) is known as the first Bianchi identity and corresponds to Eq. (40). Therefore, $\hat{T}^{\mu\nu}$ in Eq. (41) becomes isomorphic to the Einstein tensor $G^{\mu\nu}$, and its divergence vanishes. *Specifically, we can say that $\hat{T}^{\mu\nu}$ naturally fits into the geometrodynamics of general relativity.*

- (iii) Since the nonlightlike U^μ has a spacelike momentum field parametrized by κ_0 in the aforementioned spacelike KG equation, it forms a submanifold of de Sitter space (a pseudo-hypersphere \mathfrak{D} embedded in R^5) with a geometrical structure similar to the spacelike KG equation, with the radius of \mathfrak{D} corresponding to κ_0 as a scale parameter. According to Sakuma *et al.*,^{25,28} the importance of de Sitter space is twofold. First, using a spacelike momentum field in de Sitter space, Snyder³³ derived a space–time quantization with a built-in Lorentz invariance; second, de Sitter space is a solution of the Einstein field equation, which describes the accelerated expansion of the universe. In accordance with Snyder's work, they showed that the quantized form of $\hat{T}^{\mu\nu}$ can be given by a combined form of the Majorana fermion field, which behaves as an energy–momentum tensor with “virtual photons” acting as mediators of electromagnetic field interactions and $\hat{T}^{\mu\nu}$ associated with a unique ground state M_g , which can be regarded as a compound Rarita–Schwinger state with a spin of 3/2. In terms of the accelerated expansion of the universe, they considered the possibility that the trace of the energy–momentum tensor representing M_g can be interpreted as a “reduced cosmological constant” Λ_{DP} (negative in our sign convention) whose magnitude can be evaluated by the new theory of dressed photons. In fact, Λ_{DP} is $-2.47 \times 10^{-53} \text{ m}^{-2}$, while the observed value of Λ_{obs} , derived from Planck satellite observations,³⁴ is $-3.7 \times 10^{-53} \text{ m}^{-2}$.
- (iv) The light field, including the newly identified spacelike counterpart U_μ , is an essential element in the theory of dressed photons. An especially intriguing aspect of dressed photons is that even in the lightlike field U_μ that satisfies $(U_\nu)^*U^\nu = 0$, where $(U_\nu)^*$ is the complex conjugate of U_ν , the spacelike KG equation $\nabla_\sigma\nabla^\sigma\lambda - (\kappa_0)^2\lambda = 0$ is “encoded” in this lightlike field through CP. If the conformal symmetry of the lightlike U_μ field, which has $W^2 = 0$ and may be related to the Weyl curvature hypothesis proposed by Penrose,²⁷ breaks, then the spacelike U_μ field emerges, along with the above-mentioned M_g , which is responsible for generating de Sitter space. de Sitter space has the unique

structural characteristic of twin universes,²⁸ with each twin universe separated by the hypersurface of the event horizon embedded in it. Thus, starting from “the big bang” in the respective domains caused by the conformal symmetry breaking of the lightlike U_μ field, the twin universes (with one consisting of ordinary matter and the other consisting of anti-matter in the sense of time reversal) merge eons later at the event horizon and return to the original light phase. In this scenario, the creation and annihilation of the universe can be compared to those of elementary particles according to the intermediation of the light field. We believe that this cyclic twin universe cosmology is similar to the conformal cyclic cosmology³⁵ proposed by Penrose.

The newly proposed cosmology described above includes two noteworthy features that are missing from current cosmology based on cosmic inflation scenarios. First, in the former, instead of treating vacuum energy as starting from nothing, we assume the infinite cyclicity of the twin universes, with the “nodes” represented as a “lightlike universe” with null distance. One cycle begins with the conformal symmetry breaking of the nodal lightlike universe, and because of the property $W^2 = 0$ ²⁸ (the aforementioned Weyl curvature hypothesis) of the nodal universe, the isotropy of the emerging nonlightlike universe can be naturally explained, in sharp contrast to inflation scenarios. Furthermore, the twin structure of the universe provides a simple solution to the missing anti-matter problem.

5.2.3. On the flatness of the universe and the thermodynamic twin structure

In addition to the isotropy of the universe, we must consider the flatness problem. Sakuma *et al.*²⁵ proposed an SCSB hypothesis, which is referred to in the argument immediately before Eq. (38). According to this hypothesis, the twin universes are assumed to be metric space-times emerging from simultaneous transitions from null electromagnetic and gravitational fields that satisfy $(U_\nu)^*U^\nu = 0$ and $W^2 = 0$, respectively, to the symmetry breaking spacelike $(U_\nu)^*U^\nu < 0$ and $W^2 \neq 0$ fields. The key assumption of SCSB is that this transition can be parametrized by κ_0 , and our reasoning is based on the following two facts:

- (a) The abundance ratio of dark matter to dark energy is approximately 1/3.
- (b) The physical meaning of the cosmological term $\Lambda g^{\mu\nu}$ can be explained by Eq. (34).

In Subsec. 5.2.1, we discussed a new plausible form of dark matter; however, given its overwhelming presence over ordinary matter, we do not believe that the dark matter modeled by $\tilde{\Omega}u^\mu$ is the most dominant form of dark matter.

As we explained in item (iii) of Subsec. 5.2.2, in the model of Sakuma *et al.*, the observable effect of dark energy is generated by the reduced cosmological constant Λ_{DP} , which has been shown to be proportional to $3(\kappa_0)^2$. We include the factor 3 in

the expression of Λ_{DP} because the spatial dimension of our universe is three. Recall that our original goal, which led to the introduction of Λ_{DP} , was to properly evaluate the involvement of the spacelike momentum field in quantum field interactions, and the existence of Λ_{DP} was derived from electromagnetic field interactions as an important parameter characterizing the ground state of spacelike “virtual photons”. The similarity between Newtonian gravity and electromagnetic Coulomb force strongly suggests a possibility that the physical entity similar to spacelike “virtual photons” must be involved in gravitational field interactions. Thus, we believe that it is natural to assume that a similar argument can be extended to gravitational field interactions, although we do not have a satisfactory quantum gravitational theory. In electromagnetism, the spacelike extension of Maxwell theory is given by Eq. (41). For the gravitational case, based on Eq. (39), we introduce a spacelike extension with the form $\tilde{T}^{\mu\nu} = \tilde{\Omega}u^\mu u^\nu$, where u^μ satisfies the spacelike four-vector condition $u^\nu u_\nu = -1$. We investigated this form because the important scalar parameter in our dark matter model must be identified in terms of the “zero-point energy” of the $\tilde{\Omega}$ field, which exists implicitly in the quantum version of Eq. (11).

The energy-quantized version of $\tilde{T}^{\mu\nu}$ is the one in which $\tilde{\Omega}$ is discretized and the minimum value of $\tilde{\Omega}_0 := \text{Min}[\tilde{\Omega}] > 0$ exists, which can be compared to the zero-point energy $h\nu/2$ of a harmonic oscillator. As previously stated in the discussion of dark energy by Sakuma *et al.*,²⁵ the spacelike energy–momentum tensor $\tilde{T}^\mu_\nu = \tilde{\Omega}u^\mu u_\nu$ is not observable, except for its trace, which can be observed as invariant under general coordinate transformations. On the basis of their arguments, we may say that the zero-point energy $-\tilde{\Omega}_0$, as the trace of $(\tilde{T}^\mu_\nu)_0$, can be reinterpreted by transferring it from the right-hand side of the Einstein field Eq. (37) to the left-hand side as the *hypothetical cosmological term* $\Lambda(\tilde{\Omega}_0)g^{\mu\nu}$, in which the existence of $g^{\mu\nu}$ is formal and has no physical meanings, where

$$4\Lambda(\tilde{\Omega}_0) := \frac{8\pi G}{c^4} \tilde{\Omega}_0. \tag{44}$$

In contrast to the case of dark energy, however, we can investigate a different possibility in the dark matter model; that is, based on guiding fact (b), we can interpret $\Lambda(\tilde{\Omega}_0)g^{\mu\nu}$ not as the *hypothetical cosmological term* mentioned above but instead as “a real” *cosmological term* with the form $\Lambda_{\text{dm}}g^{\mu\nu}$, where $\Lambda_{\text{dm}} := \Lambda(\tilde{\Omega}_0)$ and $g^{\mu\nu}$ represent the energy–momentum tensor of the conformal gravity field given by Eq. (34). Then, by applying guiding fact (a) to $\Lambda_{\text{dm}} = \Lambda(\tilde{\Omega}_0)$, we can introduce a core assumption of the SCSB hypothesis, which has the form

$$\Lambda_{\text{dm}} = \Lambda(\tilde{\Omega}_0) = -\Lambda_{\text{DP}}/3 > 0. \tag{45}$$

First, we note that the condition of $W^2 \neq 0$ in Eq. (34) is necessary to define $g^{\mu\nu}$ as the energy–momentum tensor of the conformal gravity field, which allows us to hypothesize that Λ_{dm} is directly related to the minimum value of quantized $|W^2|$. Recall that the magnitude of $\tilde{\Omega}_0$ depends on the undetermined constant parameter T_R , which was introduced in the argument of Eq. (38). By adjusting this parameter,

we can ensure that $\Lambda(\tilde{\Omega}_0)$ is equal to Λ_{dm} , which corresponds to the minimum value of quantized $|W^2|$. Second, as previously mentioned, $-\Lambda_{\text{DP}}$ is proportional to $3(\kappa_0)^2$; thus, according to Eq. (45), the parameter Λ_{dm} of the gravitational field is related to the parameter $(\kappa_0)^2$ of the electromagnetic field. Since the factor of 3 in $-\Lambda_{\text{DP}}$ reflects the spatial dimension of our universe, we hypothesize that $\Lambda_{\text{dm}} \propto (\kappa_0)^2$ implies the “equipartition of energy” in four-dimensional space–time, which can be concisely represented by a sign convention with the form $(+, -, -, -)$. The overwhelmingly large abundance ratios of dark energy and dark matter to ordinary matter suggest that the space–time structure of our universe is determined by these two dark components. As a result of the above-mentioned “equipartition of energy”, our universe has a nearly flat space–time structure. Thus, Eq. (45) relates the parameter of conformal symmetry breaking in an electromagnetic field to that in a gravitational field in a way that it is consistent with observational evidence.

Λ_{dm} , as defined in Eq. (45), is a unique parameter of anti-de Sitter space (ADS), the theoretical importance of which can be appreciated by ADS/CFT correspondence. Although an expansion-accelerated universe, such as ours, is not ADS itself, it is worth noting that the scale of our universe as one of the pairs is given exactly by Λ_{dm} ;²⁸ thus, in this sense, ADS must be a genuine scale parameter of our universe. If we choose a natural unit system in which the speed of light c and the Planck constant h have unit magnitude, then the length scale $\sqrt{1/(\kappa_0)^2} \approx 40$ nanometer, called the dressed photon constant, can be shown to be the geometric mean²⁸ of the smallest Planck length and the largest $\sqrt{1/\Lambda_{\text{dm}}}$. It is also worth noting that $\sqrt{1/(\kappa_0)^2}$ provides a rough estimate of the Heisenberg cut for electromagnetic phenomena. We can argue that the dark matter phenomenon can be explained by combining $\tilde{\Omega}u^\mu$ currents in regions where $R^{\mu\nu}$ is negligibly small, which was noted in the argument after Eq. (39), and the result derived of Eq. (45).

Regarding the twin structure of universes and the thermodynamic aspect of space–time, we briefly refer to Tomita–Takesaki theory³⁶ on the Kubo–Martin–Schwinger (KMS) state, which can be regarded as a generalization of the Gibbs state that covers thermodynamic equilibrium states with infinite degrees of freedom for which we cannot define trace operations. Since the KMS state is a mixed state, its corresponding Gel’fand–Naimark–Segal representation is reducible. Therefore, for \mathcal{M} defined as a von Neumann algebra in Hilbert space \mathfrak{H} , there exists a commutant \mathcal{M}' that satisfies the following inversion relation:

$$J\mathcal{M}J = \mathcal{M}', \quad e^{itH}\mathcal{M}e^{-itH} = \mathcal{M}, \quad J^2 = 1, \tag{46}$$

$$JHJ = -H, \tag{47}$$

where H and J denote the Hamiltonian and the anti-unitary operator known as the modular conjugate operator. The spectrum of the Hamiltonian is symmetric with regard to its sign, indicating the existence of states with negative energy, whose stability is known to be greater than that of a vacuum state.³⁷ Thus, we believe that

the result of Tomita–Takesaki theory applies to the case of twin universes discussed in this paper.

Acknowledgments

We appreciate the anonymous reviewers for their valuable comments which helped us to improve the quality of this paper. This research was partially supported by a collaboration with the Institute of Mathematics for Industry, Kyushu University and the first author is especially thankful for the help of Prof. Y. Fukumoto in promoting this collaboration.

Appendix A

Dividing Eq. (15) by n , we have

$$(w/n)u^\nu\partial_\nu u_\mu - (\partial_\mu p)/n + (u_\mu u^\nu/n)\partial_\nu p = 0. \tag{A.1}$$

The first term on the left-hand side of (A.1) can be rewritten as

$$\begin{aligned} u^\nu(w/n)\partial_\nu u_\mu &= u^\nu[\partial_\nu\{(w/n)u_\mu\} - u_\mu\partial_\nu(w/n)] \\ &= u^\nu[\partial_\nu\{(w/n)u_\mu\} - \partial_\mu\{(w/n)u_\nu\} + \partial_\mu\{(w/n)u_\nu\}] \\ &\quad - u^\nu u_\mu\partial_\nu(w/n). \end{aligned} \tag{A.2}$$

With the new notation for the vorticity field $\omega_{\mu\nu} := \partial_\mu[(w/n)u_\nu] - \partial_\nu[(w/n)u_\mu]$, (A.2) becomes

$$u^\nu(w/n)\partial_\nu u_\mu = -\omega_{\mu\nu}u^\nu + u^\nu\partial_\mu\{(w/n)u_\nu\} - u^\nu u_\mu\partial_\nu(w/n). \tag{A.3}$$

By substituting (A.3) into (A.1), we obtain

$$-\omega_{\mu\nu}u^\nu + u^\nu\partial_\mu\{(w/n)u_\nu\} - (\partial_\mu p)/n - u_\mu u^\nu[\partial_\nu(w/n) - (\partial_\nu p)/n] = 0. \tag{A.4}$$

Next, by applying $u^\nu u_\nu = 1$ to the second term in (A.4), we obtain

$$u^\nu\partial_\mu\{(w/n)u_\nu\} = \partial_\mu\{(w/n)u^\nu u_\nu\} - (w/n)u_\nu\partial_\mu u^\nu = \partial_\mu(w/n). \tag{A.5}$$

In addition, according to Eqs. (9) and (13), the sum of the last two terms in (A.4) becomes $-u_\mu u^\nu T\partial_\nu(\sigma/n) = 0$; thus, we can finally obtain that

$$-\omega_{\mu\nu}u^\nu + \partial_\mu(w/n) - (\partial_\mu p)/n = 0 \Rightarrow \omega_{\mu\nu}u^\nu = T\partial_\mu(\sigma/n). \tag{A.6}$$

References

1. J. Bekenstein, *Lett. Nuovo Cimento* **4**, 737 (1972).
2. S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
3. W. G. Unruh, *Phys. Rev. D* **14**, 870 (1976).
4. E. Verlinde, *J. High Energy Phys.* **4**, 29 (2011).
5. R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and all that* (Benjamin, 1964), p. 165.

6. R. Jost, *The General Theory of Quantized Fields* (American Mathematical Society, Providence, 1963).
7. G. F. Dell'Antonio, *J. Math. Phys.* **2**, 759 (1961).
8. R. Penrose, *Quantum Theory and Beyond* (Cambridge University Press, 1971).
9. I. Ojima, Micro-macro duality in quantum physics, in *Proc. 3rd Int. Conf. "Stochastic Analysis: Classical and Quantum — Perspectives of White Noise Theory"*, ed. T. Hida (World Scientific, 2005), pp. 143–161. arXiv:math-ph/0502038.
10. H. Sakuma, I. Ojima, M. Ohtsu and T. Kawazoe, *J. Eur. Opt. Soc. Rapid Publ.* **17**, 28 (2021). doi:10.1186/s41476-021-00171-w.
11. S. Doplicher, R. Haag and J. E. Roberts, *Commun. Math. Phys.* **13**, 1 (1969); **15**, 173 (1969); **23**, 199 (1971); **35**, 49 (1974).
12. S. Doplicher and J. E. Roberts, *Commun. Math. Phys.* **131**, 51 (1990); *Ann. Math.* **130**, 75 (1989); *Invent. Math.* **98**, 157 (1989).
13. S. Aoki, T. Onogi and S. Yokoyama, *Int. J. Mod. Phys. A* **36**, 2150098 (2021).
14. S. Aoki, T. Onogi and S. Yokoyama, *Int. J. Mod. Phys. A* **36**, 2150201 (2021).
15. H. Ertel, *Meteorol. Z.* **59**, 277 (1942).
16. V. I. Arnol'd, *Dokl. Akad. Nauk SSSR* **162**, 975 (1965) [*Sov. Math. Dokl.* **6**, 773 (1965)] (in Russian).
17. P. J. Morrison and J. M. Greene, *Phys. Rev. Lett.* **45**, 790 (1980).
18. D. D. Holm, B. A. Kuperschmidt and C. D. Levermore, *Phys. Lett. A* **98**, 389 (1983).
19. J. E. Marsden and P. J. Morrison, *Contemp. Math.* **28**, 133 (1984).
20. Y. Kuroda, *Fluid Dyn. Res.* **5**, 273 (1990).
21. H. Sakuma and Y. Fukumoto, *Publ. RIMS Kyoto Univ.* **51**, 605 (2015).
22. L. D. Landau and E. M. Lifshitz, Relativistic fluid dynamics, *Fluid Mechanics*, 2nd edn. (Elsevier, Oxford, UK, 1987), pp. 506–508.
23. A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (Benjamin, New York, 1967).
24. J. Katz, *Proc. R. Soc. Lond. A* **391**, 415 (1984).
25. H. Sakuma, I. Ojima, M. Ohtsu and H. Ochiai, *Symmetry* **12**, 1244 (2020). doi:10.3390/sym12081244.
26. H. Sakuma and H. Ochiai, Note on the physical meaning of the cosmological term, OffShell:1909O.001.v2 (2019), <http://offshell.rodrep.org/?p=249>.
27. R. Penrose, Singularities and time-asymmetry, *General Relativity: An Einstein Centenary Survey* (Cambridge University Press, 1979), pp. 581–638.
28. H. Sakuma and I. Ojima, *Symmetry* **13**, 593 (2021). <https://doi.org/10.3390/sym13040593>.
29. N. Padhye and P. J. Morrison, *Phys. Lett. A* **219**, 287 (1996).
30. H. Saigo, *Symmetry* **13**, 1727 (2021). doi:10.3390/sym13091727.
31. K. Okamura, *Symmetry* **13**, 1183 (2021). doi:10.3390/sym13071183.
32. S. H. Lamb, *Hydrodynamics*, Clebsch's Transformation, 6th edn. (Cambridge University Press, Cambridge, UK, 1930), pp. 248–249.
33. H. S. Snyder, *Phys. Rev.* **71**, 38 (1947).
34. H. Liu, <https://www.quora.com/What-is-the-best-estimate-of-the-cosmological-constant>.
35. R. Penrose, Before the big bang: An outrageous new perspective and its implications for particle physics, in *Proc. EPAC 2006*, Edinburg, Scotland, 2006, pp. 2759–2762.
36. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics, Vol. 128 (Springer, 1970), doi:10.1007/BFb0065832, ISBN:978-3-540-04917-3.
37. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Vol. 1 (Springer-Verlag, 1979).